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Uniform Spaces and Gromov Hyperbolicity

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Abstract. This brief outline contains mostly definitions, background information, and statements of theorems. Along with the title topics, we also discuss the uniformization volume growth problem as well as certain capacity and slice condition characterizations of uniformity.

Keywords. uniform spaces, Gromov hyperbolicity, quasihyperbolic metric, volume growth, Ahlfors regular spaces, Loewner spaces, slice conditions.

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1. Introduction

This survey is meant to supplement the talks I presented at the International Workshop on Quasiconformal Mappings and their Applications and at the International Conference on Geometric Function Theory, Special Functions and their Applications. Primarily, I provide here basic background material including definitions, terminology, and fundamental facts. I also list a few references, many of which themselves contain additional references to this material. I have made no attempt to render a complete list of references and apologize to all those whose work I have neglected to mention. The reader is absolutely encouraged to consult the many works referred to by the authors I do mention.

The goal of these notes is to provide the reader with a foundation enabling them to understand the meaning and relevance of the recent work [BHK01], [BHR01], [BKR98] of Bonk, Heinonen, Koskela and Rohde along with [Her04] and [Her06]. I am delighted to thank Mario Bonk, Juha Heinonen and Pekka Koskela for numerous helpful discussions and hours of blackboard sessions regarding these topics.

By now Euclidean uniform spaces (domains in Euclidean space in which points can be joined by short twisted double cone arcs) are well recognized as being the 'nice' spaces for quasiconformal function theory as well as many other areas of analysis (e.g., potential theory); see [Geh87], [Väi88] for Euclidean space and [Gre01], [CGN00], [CT95] for the Carnot-Carathéodory setting. In [BHK01] Bonk, Heinonen, and Koskela develop a uniformization theory which provides a two way correspondence between uniform spaces and Gromov hyperbolic spaces. In particular, they prove the following fundamental result; see [BHK01, Theorem 1.1].

> *There is a one-to-one (conformal) correspondence between quasiisometry classes of proper geodesic roughly starlike Gromov hyperbolic spaces and quasisimilarity classes of bounded locally compact uniform spaces.*

A simple, yet beautiful, example is the open unit disk in the plane. In terms of its Euclidean geometry, each pair of points can be joined by a twisted double cone which stays away from the boundary and is not much bigger than the distance between the given points (it is a uniform space). On the other hand, the disk also admits a non-euclidean geometry, in terms of its Poincaré hyperbolic metric, and as such the disk is a Gromov hyperbolic space.

The Bonk, Heinonen, Koskela theory asserts that this phenomenon holds in a very general setting. The complete proof of their result is presented in Chapters 2-5 of [BHK01] and beyond the scope of our discussion. However, there are two basic results involved which are central to my workshop lectures: Fact 4.1 says that every locally compact uniform space has a Gromov hyperbolic quasihyperbolization; Fact 5.1 says that every (proper geodesic) Gromov hyperbolic space can be uniformized. I will describe what uniform spaces are, what their connection is with Gromov hyperbolicity, and explain some of the ideas behind the proofs. Time permitting, I will also look at the related question of when there exists a uniformization with the property that the associated measure (see (2.7)) has regular volume growth. My conference lecture will focus on §6.D and §6.E.

For the remainder of this introduction, I advertise results from [Her04] and [Her06] hoping to wet the reader's appetite for this flavor of metric measure space geometric function theory. See §2-§5 for precise definitions.

In [BHR01] and [BKR98] the authors investigate conformal deformations of the unit ball in Euclidean space. The primary object of study in these notes is the geometry of quasiconformal deformations of an abstract metric measure space (Ω, d, μ). Following BHKR, we consider a metric-density ρ on Ω and $\Omega_\rho = (\Omega, d_\rho, \mu_\rho)$ denotes the deformed space (see subsection 2.G). We are interested in the situation when this new space Ω_ρ is uniform (see Section 3) and describe this by calling such a ρ a *uniformizing density*. Every proper geodesic Gromov hyperbolic space can be uniformized, and, there is a natural canonical proper geodesic space associated with any locally compact abstract domain, namely, its quasihyperbolization; see Facts 4.1 and 5.1. However, in general the associated measure (see (2.7)) may fail to have Ahlfors regular volume growth. For example, applying the BHK uniformization to the quasihyperbolized Euclidean unit ball we obtain a new metric measure space which has exponential volume growth.

The theory developed in [BHK01] is exploited in [Her06] to extend some results of [BHR01] to the setting of abstract metric measure spaces (Ω, d, μ). More importantly, we establish the result given below which provides an answer to the question: When does an abstract domain admit a quasiconformal deformation which is both uniformizing and such that the induced measure (2.7) satisfies the natural volume growth estimate? That is, when is there a conformal uniformizing density? In particular, the induced measure should be Ahlfors regular. Under certain reasonable minimal hypotheses, this occurs precisely when the conformal Assouad dimension of the space's Gromov boundary is small enough. See §5.E for a discussion of the proof of the following.

Theorem A. *Let Ω be an abstract domain with bounded Q-geometry. Suppose Ω admits a bounded uniformizing conformal density. Then Ω has a Gromov hyperbolic roughly starlike quasihyperbolization and the conformal Assouad dimension of its Gromov boundary is strictly less than Q. The converse holds too, provided we assume that the Gromov boundary of Ω is uniformly perfect.*

The above result is quantitative: the asserted constants depend only on the data associated with Ω and the density.

In what follows we consider metric measure spaces (Ω, d, μ) which satisfy the following *basic minimal hypotheses*:

> Ω is an abstract domain having bounded Q-geometry and a
> Gromov hyperbolic roughly starlike quasihyperbolization.

Precise definitions are stated in subsections 2.B, 2.D, 2.H, 5.C; roughly, these hypotheses ensure that Ω has 'enough' of the local properties enjoyed by domains in Euclidean space. The *data* associated with these basic hypotheses consists of six parameters: Q (the 'dimension'), M, m, λ (the bounded geometry constants), δ (the Gromov hyperbolicity constant) and κ (the rough starlike constant).

There are a number of auxiliary results (namely, Theorems B-F) needed for the proof of Theorem A; all of these can be found in [Her04] or [Her06]. First we have the so-called Gehring-Hayman Inequality (cf. [GH62]); it is an essential tool for most of what follows. This was proved in [BKR98, Theorem 3.1] for deformations of the Euclidean unit ball and in [HR93] for quasiconformal images of uniform domains in Euclidean space; see also [BB03, Theorem 2.3], [BHK01, Chpt. 5] and [HN94]. Our proof of the following (see [Her04, Theorem A]) utilizes ideas from both [BKR98, Theorem 3.1] and [HR93, Theorem 1.1].

Theorem B. *Let ρ be an Ahlfors Harnack density on a uniform Loewner metric measure space (Ω, d, μ). Then there exists a constant Λ such that for all quasihyperbolic geodesics $[x, y]_k$ with endpoints in $\bar{\Omega}$,*

$$\ell_\rho\left([x, y]_k\right) \leq \Lambda\, d_\rho(x, y).$$

This result is quantitative: Λ depends only on the data associated with Ω. Throughout this article the symbol Λ will stand for this Gehring-Hayman Inequality constant.

Here is a simple, but useful, consequence of the Gehring-Hayman Inequality: if there is an arc α joining some point w in Ω to some point ζ in $\partial\Omega$ with

$\ell_\rho(\alpha) < \infty$, then $\ell_\rho(\gamma) < \infty$ for every quasihyperbolic geodesic ray going to ζ. In fact, there is even a 'radial limit theorem' [Her04, Theorem B] which says that this is true for mod_Q-a.e. point of $\partial\Omega$.

Next we communicate the primary tool employed in our proof of Theorem A. It is based on a lifting procedure discussed in [BKR98, 2.7] and established for the Euclidean unit ball as [BHR01, Proposition 1.25]. See (5.9) and (2.5) for the definitions of ρ_ν (the *lift* of ν) and $\delta_{\nu,1/P}$ (the quasimetric determined by ν). See §5.D for a discussion of the proof of the following.

Theorem C. *Assume the basic minimal hypotheses, that the Gromov boundary of Ω is uniformly perfect, and that $P < Q$. Suppose ν is a P-dimensional metric doubling measure on $\partial_G\Omega$. Then the lift $\rho = \rho_\nu$ of ν is a doubling conformal density on Ω and the natural map $(\partial_\rho\Omega, d_\rho) \to (\partial_G\Omega, \delta_{\nu,1/P})$ is bilipschitz.*

The Bonk-Heinonen-Koskela uniformization theory is a crucial tool employed in all our arguments and permits us to replace the space Ω with a bounded uniform space Ω_ε where the geometry is more transparent; see Fact 5.1. A key ingredient in our proof of Theorem A is the following generalization of [BHR01, Proposition 2.11]. In particular, it asserts that a conformal density on a bounded uniform space is uniformizing if and only if the associated measure (2.7) is a doubling measure on the original space. (See §5.E for the precise definition of a doubling conformal density.)

Theorem D. *Assume the basic minimal hypotheses. Let Ω_ε be any BHK-uniformization of Ω. Suppose ρ is a conformal density on Ω. Then the following are quantitatively equivalent:*

(a) *ρ is doubling on Ω.*
(b) *Ω_ρ is bounded and uniform.*
(c) *Ω_ρ is bounded and Q-Loewner.*
(d) *Ω_ρ is bounded, Q-Loewner and Ahlfors Q-regular.*
(e) *the identity map $\Omega_\rho \to \Omega_\varepsilon$ is quasisymmetric.*

Again, this result is quantitative: the asserted constants depend only on the data associated with Ω and ρ, and the related data. Also, we point out that the proof of (b) shows that the quasihyperbolic geodesics in Ω will be uniform arcs in Ω_ρ.

A crucial component of the proof of Theorem C is the following result which permits us to estimate $d_\rho(x) = \mathrm{dist}_\rho(x, \partial_\rho\Omega)$ in terms of $\rho(x)d(x)$. More precisely, it tells us that Ahlfors Harnack metric-densities are Koebe under the right conditions. The lower bound is immediate via the Harnack inequality. To obtain any upper bound, we at least need $\partial_\rho\Omega \neq \emptyset$. In fact, we require a condition which ensures that Ω has a uniformly thick boundary as seen from each point. With this in mind, we introduce the following notion: we say that (Ω, d, μ) satisfies a *Whitney ball modulus property* if there exists a constant $m > 0$ such that

$$\mathrm{mod}_Q(\bar{B}(x; \lambda\, d(x)), \partial\Omega; \Omega) \geq m \qquad \text{for all } x \in \Omega.$$

Theorem E. *Let ρ be a Ahlfors Harnack metric-density on a uniform Loewner abstract domain (Ω, d, μ). Suppose that Ω enjoys a Whitney ball modulus property. Then there is a constant K such that for all $x \in \Omega$,*

$$K^{-1}\rho(x)d(x) \leq d_\rho(x) \leq K\rho(x)d(x);$$

the constant K depends only on the data associated with Ω.

An important consequence of Theorem E is that the quasihyperbolizations of Ω and Ω_ρ are bilipschitz equivalent, and it follows that (Ω_ρ, k_ρ) is a Gromov hyperbolic space.

We mention that any uniform Loewner space with connected boundary satisfies a Whitney ball modulus property, provided it and its boundary are simultaneously bounded or unbounded. Similarly any bounded uniform Loewner space with a finite number of non-degenerate boundary components will enjoy this modulus property. Here is a sufficient condition for this property to hold which allows for a totally disconnected boundary.

Theorem F. *Let (Ω, d, μ) be a locally Loewner, uniform metric measure space. Assume Ω and $\partial\Omega$ are either both bounded or both unbounded. Suppose that for some $p > 0$, $\partial\Omega$ satisfies the Hausdorff p-content condition*

$$\mathcal{H}^p_\infty(\partial\Omega \cap \bar{B}(\zeta; r)) \geq c\, r^p \qquad \text{for all } 0 < r \leq \operatorname{diam}(\partial\Omega) \text{ and all } \zeta \in \partial\Omega.$$

Then Ω enjoys a Whitney ball modulus property with a constant m which depends only on c and the data associated with Ω.

In contrast to the Euclidean case, the converse to the above is false; see [Her04, Example 3.2] which furnishes a space with an isolated boundary point which nonetheless satisfies a Whitney ball modulus property.

Our notation is relatively standard and, for the most part, conforms with that of [BHK01]. We write $C = C(a, \ldots)$ to indicate a constant C which depends only on the parameters $a, \ldots$; the notation $A \lesssim B$ means there exists a finite constant c with $A \leq cB$, and $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. Typically $a, b, c, C, K, \ldots$ will be constants that depend on various parameters, and we try to make this as clear as possible often giving explicit values, however, at times C will denote some constant whose value depends only on the data present but may differ even on the same line of inequalities.

2. Metric Space Background

Naturally there are scores of references for metric space geometry. Here is a brief list of some texts which I have found especially helpful: [BH99], [BBI01], [Hei01], [Sem01], [Sem99], [DS97], and of course the references mentioned in these works.

2.A. General Information. In what follows (X,d) will always denote a generic metric space possessing no additional presumed properties. For the record, this means that d is a distance function; that is, $d : X \times X \to \mathbb{R}$ is positive semi-definite, symmetric, and satisfies the triangle inequality. We often write the distance between x and y as $d(x,y) = |x-y|$. The open ball (sphere) of radius r centered at the point x is $B(x;r) := \{y : |x-y| < r\}$ $(S(x;r) := \{y : |x-y| = r\})$. When $B = B(x;r)$ and $\lambda > 0$, $\lambda B := B(x;\lambda r)$. We say that X is a *proper* metric space if it has the Heine-Borel property that every closed ball is compact (or equivalently, the compact sets are exactly the closed and bounded sets).

In general, we work in the setting of a metric measure space (X,d,μ) with X a non-complete locally complete (often locally compact) rectifiably connected metric space and μ a Borel regular measure satisfying $\mu[B(x;r)] > 0$ for each ball.

Recall that every metric space can be isometrically embedded into a complete metric space. We let $\bar{X}$ denote the metric completion of a metric space X and we call $\partial X = \bar{X} \setminus X$ the metric boundary of X. Then $d(x) = \operatorname{dist}(x, \partial X)$ is the distance from a point $x \in X$ to the boundary ∂X of X; note that when ∂X is closed in $\bar{X}$, we have $d(x) > 0$ for all $x \in X$. For example, this holds when X is locally compact. Of course, if X is complete to begin with, then $\partial X = \emptyset$ and $d(x) = \infty$ for all $x \in X$. We call X *locally complete* provided $d(x) > 0$ for all $x \in X$.

In a locally complete metric space we make extensive use of the notation

$$B(x) := B(x; d(x)).$$

In this setting, we call $\lambda B(x) = B(x; \lambda\, d(x))$ a *Whitney ball* in X with associated *Whitney ball constant* $\lambda \in (0,1)$.

It is convenient, at times, to consider *quasimetric* spaces (X,q). We call q a *quasimetric* on X if $q : X \times X \to \mathbb{R}$ is symmetric and positive definite but only satisfies

$$q(x,y) \le K\,(q(x,z) + q(y,z)) \qquad \text{for all } x,y,z \in X$$

in place of the triangle inequality. See [Hei01, 14.1], [Sem01] and [DS97].

Starting with a quasimetric q, there is a standard way to define a pseudometric d with $d \le q$ (cf. [BH99, 1.24, p.14]), but it may happen that $d(x,y) = 0$ for some $x \ne y$. However, by first 'snowflaking' q and then applying this procedure we can arrive at an honest distance function; see [Hei01, Proposition 14.5] or [BH99, Proposition 3.21, p.435].

2.1. Fact. *Let q be a quasimetric on X. There is an $\varepsilon_0 > 0$ depending only on the quasimetric constant K for q such that for all $\varepsilon \in (0,\varepsilon_0)$, the quasimetric $q_\varepsilon(x,y) = q(x,y)^\varepsilon$ is bilipschitz equivalent to an honest distance function d on X; in fact there is a constant $L = L(\varepsilon, K)$ such that*

$$L^{-1} q_\varepsilon(x,y) \le d(x,y) \le q_\varepsilon(x,y) \qquad \textit{for all } x,y \in X.$$

We remark that all the quasimetrics q_ε as defined above are QS equivalent to each other.

Another useful notion, apparently introduced by Väisälä, is that of a *metametric* $m : X \times X \to \mathbb{R}$ which is symmetric, non-negative, satisfies the triangle inequality, but only

$$m(x, y) = 0 \implies x = y$$

and so possibly $m(x, x) > 0$. See [Väi05a, 4.2] for a treatment of metametric spaces.

A metric space X is called *uniformly perfect* provided it has at least two points and there is a constant $\vartheta \in (0, 1)$ such that for all balls $B \subset X$, $B \setminus \vartheta B \neq \emptyset$ provided $X \setminus B \neq \emptyset$. This concept, which involves three points, is especially useful when dealing with quasisymmetric maps and also with doubling measures (see §2.F). The property of being uniformly perfect is preserved by quasisymmetric homeomorphisms, with the new constant depending only on the original constant and the quasisymmetry data; in particular, one can ask whether or not a conformal gauge is uniformly perfect (see §2.C).

It is a routine exercise to see that uniformly perfect locally compact spaces contain quasisymmetrically embedded middle-third Cantor dusts. Using this fact, together with a scaling argument and properties of quasisymmetric homeomorphisms (e.g. [Hei01, 11.10,11.11]), one can verify a version of the following. For a simple more direct approach, which also provides the indicated explicit constants, see [Her06, Lemma 4.2].

2.2. Fact. *Suppose X is a uniformly perfect compact metric space. Then X satisfies the p-dimensional Hausdorff measure density condition*

$$\mathcal{H}^p[B(x; r)] \geq \frac{r^p}{6} \quad \textit{for all } 0 < r \leq \operatorname{diam}(X) \textit{ and all } x \in X,$$

where $p = 1/\log_2(4/\vartheta)$ and ϑ is the uniform perfectedness constant.

The above result can be used in conjunction with Theorem F to see that the Whitney ball modulus property holds.

2.B. Abstract Domains. We call a metric measure space (Ω, d, μ) an *abstract domain* if Ω is a non-complete locally complete rectifiably connected metric space (and μ a Borel regular measure with dense support). An important example of such a space is, of course, a proper subdomain of Euclidean space with either Euclidean distance or the induced Euclidean length distance.

Unless explicitly indicated otherwise, the adjective *locally* means that the modified property or condition holds in all Whitney-type balls $\lambda B(x)$ where $0 < \lambda < 1$ is some fixed constant which we call the *Whitney ball constant*; when there are several such local conditions in play, we always take λ to be the minimum of all the associated Whitney ball constants.

2.C. Maps and Gauges. An embedding $f : X \to Y$ from a metric space X into a metric space Y is *quasisymmetric*, abbreviated QS, if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ (called a *distortion function*) such that for all triples $x, y, z \in X$,

$$|x-y| \le t|x-z| \implies |fx-fy| \le \eta(t)|fx-fz|.$$

These mappings were studied by Tukia and Väisälä in [TV80]; see also [Hei01]. The *bilipschitz* maps form an important subclass of the quasisymmetric maps; $f : X \to Y$ is bilipschitz if there is a constant L such that for all $x, y \in X$,

$$L^{-1}|x-y| \le |fx-fy| \le L|x-y|.$$

More generally, a map $f : X \to Y$ is an (L, C)-*quasiisometry* if $L \ge 1$, $C \ge 0$ and for all $x, y \in X$,

$$L^{-1}|x-y| - C \le |fx-fy| \le L|x-y| + C.$$

There seems to be no universal agreement regarding this terminology; some authors use the adjective quasiisometry to mean what we have called bilipschitz, and then a rough quasiisometry satisfies our definition of quasiisometry. So the reader should beware! Of course a $(1, 0)$-quasiisometry is simply called an *isometry* (onto its range).

Note that the above definitions also make sense for mappings of quasimetric spaces.

Given a metric (or a quasimetric) on X, we can form the *conformal gauge* $\mathcal{G}$ on X consisting of all metrics on X which are QS equivalent to the original (quasi)metric. That is, $\mathcal{G}$ is the family of all metrics ∂ on X such that the identity map $(X, d) \to (X, \partial)$ is QS. See [Hei01, Chapter 15] for more discussion of this topic.

An embedding $f : X \to Y$ from a metric space X into a metric space Y is called *quasimöbius*, abbreviated QM, if there is a homeomorphism $\vartheta : [0,\infty) \to [0,\infty)$ (called a *distortion function*) such that for all quadruples x, y, z, w of distinct points in X,

$$|x, y, z, w| \le t \implies |fx, fy, fz, fw| \le \vartheta(t)$$

where the *absolute cross ratio* is

$$|x, y, z, w| = \frac{|x-y||z-w|}{|x-z||y-w|}.$$

These mappings were introduced and investigated by Väisälä in [Väi85]; see also [Väi05a]. Every QS homeomorphism is QM; the converse holds in certain special cases. Clearly Möbius transformations are QM maps in Euclidean space; however, a Möbius transformation from the unit ball onto a half-space is not QS. The QM maps are more flexible than the QS.

The QS and QM maps are defined by global conditions whereas QC (quasiconformal) maps only satisfy a local condition. I highly recommend Tyson's recent survey article [Tys03]. Väisälä's notes [Väi71] are the classical reference for QC maps in the Euclidean setting. These maps have been studied in the Heisenberg group setting and there is still much research underway there. Heinonen and Koskela strongly advanced the theory in the general metric space setting; see [HK95] and [HK98]. See Koskela's notes [Kos07] for a 'modern' approach to QC maps in the Euclidean setting.

There are three so-called definitions for QC maps: the metric definition, the geometric definition, and the analytic definition. We present the first two. A homeomorphism $f : X \to Y$ is *(metrically) quasiconformal* provided there is a constant $H < \infty$ such that for all $x \in X$,

$$\limsup_{r\searrow 0} H(x,f,r) \leq H \quad \text{where} \quad H(x,f,r) = \frac{L(x,f,r)}{l(x,f,r)}\,,$$
$$L(x,f,r) = \sup\{|f(y)-f(x)| : |x-y| \leq r\}\,,$$
$$l(x,f,r) = \inf\{|f(y)-f(x)| : |x-y| \geq r\}\,.$$

A homeomorphism $f : X \to Y$ is *(geometrically) quasiconformal* provided there is a constant $K < \infty$ such that for all curve families Γ in X,

$$K^{-1}\,\mathrm{mod}(\Gamma) \leq \mathrm{mod}(f\Gamma) \leq K\,\mathrm{mod}(\Gamma).$$

Notice that unlike the metric definition, which makes sense for any pair of metric spaces, the geometric definition requires measure metric spaces. These are generally assumed to be Ahlfors Q-regular spaces (see §2.J) in which case $\mathrm{mod}(\cdot)$ denotes the Q-modulus.

2.D. Length and Geodesics. The length of a continuous path $\gamma : [0,1] \to X$ is defined in the usual way by

$$\ell(\gamma) := \sup\sum_{i=1}^{n}|\gamma(t_i)-\gamma(t_{i-1})| \qquad \text{where } 0 = t_0 < t_1 < \cdots < t_n = 1.$$

We call γ *rectifiable* when $\ell(\gamma) < \infty$. We let $\Gamma(x,y) = \Gamma(x,y;X)$ denote the collection of all rectifiable paths joining x and y in X; in general we should also indicate the metric in this notation, but it will always be understood from context. Väisälä's notes [Väi71, §1-§5] provide an excellent reference for studying properties of curves, and the results are valid in the general metric space setting. Each rectifiable path $\gamma : [0,1] \to X$ has an associated arclength function $s : [0,1] \to [0,\ell(\gamma)]$, given by $s(t) = \ell(\gamma[0,t])$, which is of bounded variation. Given a Borel measurable function $\rho : X \to [0,\infty]$, we define

$$\int_\gamma \rho\,ds := \int_0^1 \rho(\gamma(t))\,ds(t).$$

An *arc* in a metric space X is the homeomorphic image of an interval $I \subset \mathbb{R}$. Given two points x and y on an arc α, we write $\alpha[x,y]$ to denote the subarc of α joining x and y.

A *geodesic* in X is the image $\varphi(I)$ of some isometric embedding $\varphi : I \to X$ where $I \subset \mathbb{R}$ is an interval; we use the adjectives *segment, ray,* or *line* (respectively) to indicate that I is bounded, semi-infinite, or all of $\mathbb{R}$. When φ is

L-bilipschitz we call $\varphi(I)$ an *L-quasigeodesic*. More generally, if φ is (L,C)-quasiisometric, then we call $\varphi(I)$ an *(L,C)-quasigeodesic*. Thus γ is an L-quasigeodesic precisely when

$$\forall x,y \in \gamma : \quad \ell(\gamma[x,y]) \leq L|x-y|;$$

classically, such curves in the plane $\mathbb{R}^2$ were called *chord arc* curves.

A metric space is *geodesic* if each pair of points can be joined by a geodesic segment. We use the notation $[x,y]$ to mean a (not necessarily unique) geodesic segment joining points x,y; such geodesics always exist if our space is geodesic, but may not be unique. (If there is some other distance function, such as k, then we write $[x,y]_k$ to denote a k-geodesic joining x,y). We consider a given geodesic $[x,y]$ as being ordered from x to y (so we can use phrases such as the 'first' point encountered). An unbounded metric space is *roughly κ-starlike* with respect to a base point w if each point lies within distance κ of some geodesic ray emanating from w.

The *geodesic boundary* $\partial_{\mathrm{g}}X$ of an unbounded geodesic metric space X is the set of equivalence classes of geodesic rays in X where two such rays are considered equivalent when they are at a finite Hausdorff distance from each other. Equivalently, if $\alpha, \beta : [0,\infty) \to X$ are geodesic rays in X, then $\alpha \simeq \beta$ if $\sup_t |\alpha(t) - \beta(t)| < \infty$. The geodesic boundary of $\mathbb{R}^n$ is the sphere $\mathbb{S}^{n-1}$. The geodesic boundary of hyperbolic n-space $(\mathbb{B}^n, h)$ is also the sphere $\mathbb{S}^{n-1}$.

Every metric space (X,d) admits a natural (or *intrinsic*) metric, the so-called *length distance* given by

$$l(x,y) := \inf\{\ell(\gamma) : \gamma \text{ a rectifiable curve joining } x,y \text{ in } \Omega\}.$$

A metric space (X,d) is a *length* space provided $d(x,y) = l(x,y)$ for all points $x,y \in X$; it is also common to call such a d an *intrinsic* distance function. Notice that an l-geodesic $[x,y]_l$ is a shortest curve joining x and y.

The Hopf-Rinow Theorem (see [Gro99, p.9], [BBI01, p.51], [BH99, p.35]) says that every locally compact length space is proper (and therefore geodesic). In a general length space, when geodesics may not exist, one works with so-called short arcs; see [Väi05a].

Since $|x-y| \leq \ell(x,y)$ for all x,y, the identity map $(X,l) \xrightarrow{\mathrm{id}} (X,d)$ is Lipschitz continuous. It is important to know when this map will be a homeomorphism (cf. [BHK01, Lemma A.4, p.92]). Notice that the identity map $(X,d) \to (X,l)$ is uniformly locally Lipschitz when X is locally quasiconvex; see §2.E. More generally, one can show that the identity map $(X,l) \to (X,d)$ is a homeomorphism precisely when X satisfies a weak notion of local quasiconvexity; see [BH07].

2.E. Connectivity Conditions. A metric space (X,d) is *a-quasiconvex* provided each pair of points can be joined by a path whose length is at most a times the distance between its endpoints. A locally complete space X is *locally quasiconvex* if there exists a constant $a \geq 1$ such that for all $z \in X$, points $x,y \in \lambda B(z)$ can be joined by a rectifiable arc α in X with $\ell(\alpha) \leq a|x-y|$; we abbreviate this by the phrase 'X is locally a-quasiconvex'. (Here it is understood that there is some Whitney ball constant $\lambda \in (0,1)$ which may also depend on other parameters).

A space (X,d) is *c-linearly locally connected*, or *c-LLC*, if $c \geq 1$ and the following conditions hold for all $x \in X$ and all $r > 0$:

$$\text{(LLC}_1\text{)} \qquad \text{points in } B(x;r) \text{ can be joined in } B(x;c\,r)$$

and

$$\text{(LLC}_2\text{)} \qquad \text{points in } X \setminus \bar{B}(x;r) \text{ can be joined in } X \setminus \bar{B}(x;r/c).$$

Here the phrase 'can be joined' means 'can be joined by a continuum'. We also use the term *LLC with respect to arcs* in which case 'can be joined' means 'can be joined by a rectifiable arc'. Note that quasiconvexity implies LLC_1 (even with respect to arcs).

The generic example of a space which does not satisfy the LLC_2 condition is the interior of an infinite Euclidean cylinder such as $\mathbb{B}^{n-1} \times \mathbb{R} \subset \mathbb{R}^n$. However, for $2 \leq k < n$ the regions $\mathbb{B}^{n-k} \times \mathbb{R}^k \subset \mathbb{R}^n$ are easily seen to be 1-LLC_2. The complement of a semi-infinite slab (e.g., $\mathbb{R}^n \setminus \{(x_1,\ldots,x_n) : x_1 \geq 0, |x_n| \leq 1\}$) fails to be LLC_1.

Ahlfors regular Loewner spaces are LLC; see [HK98, Theorem 3.13]. Uniform domains also enjoy this property, but not necessarily uniform spaces. The LLC condition was invented by Gehring who first used it to characterize quasidisks; see [Geh82] and the references mentioned therein.

2.F. Doubling and Dimensions. The *p-dimensional Hausdorff measure* of a set $A \subset X$ is given by $\mathcal{H}^p(A) := \lim_{r\to 0} \mathcal{H}^p_r(A)$ where

$$\mathcal{H}^p_r(A) := \inf\{\sum \operatorname{diam}(B_i)^p : A \subset \cup B_i, B_i \text{ balls with } \operatorname{diam}(B_i) \leq r\}.$$

The *Hausdorff p-content* of A is just $\mathcal{H}^p_\infty(A)$. The *Hausdorff dimension* of A is determined by

$$\dim_{\mathcal{H}}(A) := \inf\{p > 0 : \mathcal{H}^p(A) = 0\}.$$

We also require the *Assouad dimension* of X which is given by

$$\begin{aligned} \dim_{\mathcal{A}}(X) := \inf\{p : \#S \leq C(R/r)^p \text{ for all } S \subset X \\ \text{with } r \leq |x-y| \leq R \text{ for all } x,y \in S\} \end{aligned}$$

where $\#S$ denotes the cardinality of the set S. See [Hei01, 10.15] or [Luu98, 3.2]. The spaces with finite Assouad dimension are precisely the doubling spaces (which we discuss below in more detail). Finally, the *conformal Assouad dimension* of a metric space X is

$$\text{c-}\dim_{\mathcal{A}}(X) := \inf\{\dim_{\mathcal{A}}(X,d) : d \in \mathcal{G}\},$$

where $\mathcal{G}$ is the conformal gauge on X determined by the original metric; see [Hei01, 15.8, p.125].

A metric space (X, d) satisfies a (metric) *doubling condition* if there is a constant N such that each ball in X of radius R can be covered by at most N balls of radius $R/2$; these are precisely the spaces of finite Assouad dimension. A Borel measure ν is a *doubling measure* on X if there is a constant $D = D_\nu$ such that

$$\nu[B(x; 2r)] \leq D\,\nu[B(x; r)] \quad \text{for all } x \in X \text{ and all } r > 0.$$

A Borel measure ν on X is *p-homogeneous* if there is a constant $C = C_\nu$ such that

$$\frac{\nu[B(x; R)]}{\nu[B(x; r)]} \leq C \left(\frac{R}{r}\right)^p \quad \text{for all } x \in X \text{ and all } 0 < r \leq R.$$

Obviously every homogeneous measure is doubling; the converse holds too with $C = D$ and $p = \log_2(D)$. Every Ahlfors Q-regular measure is *Q-homogeneous*.

The existence of a doubling measure is easily seen to imply a metric doubling condition; the converse holds if our metric space is complete. Here is a precise statement of this result, which is due to Vol'berg and Konyagin for compact spaces, and Luukkainen and Saksman for complete spaces (see [Hei01, Theorem 13.5]).

2.3. Fact. *A complete doubling space X carries a p-homogeneous measure for each $p > \dim_{\mathcal{A}}(X)$.*

An especially important property of doubling measures is their exponential decay on uniformly perfect spaces, which we record as follows; see [Hei01, (13.2)] or [Sem99, Lemma B.4.7, p.420].

2.4. Fact. *Let ν be a doubling measure on a uniformly perfect metric space. There are constants $C \geq 1$ and $\alpha > 0$, depending only on the doubling constant for ν and the uniformly perfect constant, such that for all balls $B(z; r) \subset B(x; R)$,*

$$\frac{\nu[B(z; r)]}{\nu[B(x; R)]} \leq C \left(\frac{r}{R}\right)^\alpha.$$

Now we discuss an interesting way to deform the geometry of a doubling space. Let ν be a doubling measure on a metric space (X, d). For each $\alpha > 0$ we define $\delta = \delta_{\nu,\alpha}$ by

$$\delta(x, y) := \nu[B(xy)]^\alpha, \quad \text{where } B(xy) := \bar{B}(x; |x - y|) \cup \bar{B}(y; |x - y|); \tag{2.5}$$

see [DS97, §16.2], [Sem99, (B.3.6)], [Hei01, 14.11]. This always defines a quasimetric on X, and, when X is uniformly perfect, the identity map $(X, d) \to (X, \delta)$ will be quasisymmetric and (X, δ, ν) will be Ahlfors $(1/\alpha)$-regular. Moreover, there is an $\alpha_0 > 0$ (depending only on the doubling constant for ν) such that for all $0 < \alpha < \alpha_0$, $\delta_{\nu,\alpha}$ is bilipschitz equivalent to an honest distance function on X (see Fact 2.1). In particular, if ν is p-homogeneous, then $\delta_{\nu,1/p}$ is already bilipschitz equivalent to an honest distance function (e.g., if (X, d, ν) is Ahlfors p-regular, then $\delta_{\nu,1/p}$ is bilipschitz equivalent to d).

In conjunction with the above chain of ideas, we declare ν to be a *p-dimensional metric doubling measure* on X if ν is a doubling measure on X with the property that $\delta_{\nu,1/p}$ is bilipschitz equivalent to a distance on X. For example, a

p-homogeneous measure will be a p-dimensional metric doubling measure. We summarize the above comments; see [Hei01, 14.11,14.14], [Sem99, B.3.7, B.4.6, p.421], [DS97, 16.5,16.7,16.8].

2.6. Fact. *Let ν be a p-dimensional metric doubling measure on a uniformly perfect metric space (X, d). Define $\delta = \delta_{\nu,1/p}$ as in (2.5). Then δ is a quasimetric on X which is bilipschitz equivalent to a distance function on X, the identity map $(X, d) \to (X, \delta)$ is quasisymmetric, and (X, δ, ν) is an Ahlfors p-regular space. All of the new parameters depend only on the original data for X and ν.*

There is one final comment we wish to point out regarding the quasimetrics $\delta_{\nu,\alpha}$. As above, suppose ν is a doubling measure on a metric space (X, d), and suppose X has another metric, say, ∂ which is QS equivalent to d. Then by using the doubling property of ν in conjunction with quasisymmetry we see that $\nu[B_d(xy)] \simeq \nu[B_\partial(xy)]$ (where these sets are defined as above using balls centered at x and y in the appropriate metrics); here the constant depends only on the doubling constant and the quasisymmetry data. It therefore follows that the quasimetric δ_d (defined as in (2.5) via $B_d(xy)$) is bilipschitz equivalent to δ_∂ (defined via $B_\partial(xy)$).

We note the important fact that quasisymmetric homeomorphisms preserve these doubling conditions; cf. [Hei01, Theorem 10.18] or [DS97, Lemma 16.4]. In particular, the notions of doubling measure, the quasimetrics $\delta_{\nu,\alpha}$, and metric doubling measures do not depend on the given distance function per se; they all make sense for a conformal gauge.

2.G. Quasiconformal Deformations. Given an abstract domain (Ω, d, μ) and a positive Borel measurable function ρ on Ω, we wish to define a new metric measure space $\Omega_\rho = (\Omega, d_\rho, \mu_\rho)$ which is a quasiconformal deformation of Ω. (Above in §2.F we described another method for deforming the geometry of Ω which was based on having a doubling measure. See Fact 2.6.)

We start by defining the ρ-length of a rectifiable curve γ via

$$\ell_\rho(\gamma) := \int_\gamma \rho\, ds$$

and then the ρ-distance between two points x, y is

$$d_\rho(x, y) := \inf\{\ell_\rho(\gamma) : \gamma \text{ a rectifiable curve joining } x, y \text{ in } \Omega\};$$

see §2.D. The careful reader no doubt recognizes that, in general, $d_\rho(x, y)$ could be zero or even infinite; in order to ensure that d_ρ be an honest distance function, we must require that $0 < d_\rho(x, y) < \infty$ for all points $x, y \in \Omega$. We designate this by calling such ρ a *metric-density* on Ω. One way to guarantee this is to ask that ρ be locally bounded away from zero and infinity. In practice, our densities will always satisfy a Harnack inequality—see below—so this is never a problem for us.

The ρ-balls (etc.) are written as $B_\rho(x;r)$; these are the metric balls in Ω_ρ, so $B_\rho(x;r) = \{y \in \Omega : d_\rho(x,y) < r\}$. We define a new measure μ_ρ by

$$\mu_\rho(E) := \int_E \rho^Q \, d\mu. \tag{2.7}$$

Here Q is usually the Hausdorff dimension of (Ω, d).

When Ω_ρ is non-complete (which will often be the case for us), we can form $\partial_\rho\Omega = \bar{\Omega}_\rho \setminus \Omega_\rho$ and define $d_\rho(x) = \mathrm{dist}_\rho(x, \partial_\rho\Omega)$. In this setting we also employ the notation $B_\rho(x) = B_\rho(x; d_\rho(x))$; thus $\lambda B_\rho(x)$ is a Whitney ball in Ω_ρ.

We are especially interested in the metric-densities ρ for which Ω_ρ is a uniform space, and we call such a ρ a *uniformizing density* (which implicitly includes the hypothesis that Ω_ρ is non-complete). The Bonk-Heinonen-Koskela theory produces uniformizing densities on proper geodesic Gromov hyperbolic spaces; see Fact 4.1. Some other classes of metric-densities which we wish to single out for attention include Harnack, Ahlfors, and Koebe densities; their definitions follow below. We let H_ρ, A_ρ, K_ρ denote the parameters associated with these densities.

Before delving into the technical definitions, we wish to make a few comments. The reader no doubt has encountered deformations of Euclidean domains $\Omega \subset \mathbb{R}^n$ by continuous densities ρ; in this setting Ω_ρ is a conformal deformation of Ω, meaning that the identify map $\Omega \to \Omega_\rho$ is conformal (i.e., metrically 1-quasiconformal). However, in our more general setting, even for the case $\rho = 1$ say, the identity map $\Omega \to \Omega_\rho = \Omega_l$ may fail to be quasiconformal (e.g., if Ω does not satisfy some sort of local quasiconvexity condition). A similar phenomenon holds for Borel metric-densities, even for domains $\Omega \subset \mathbb{R}^n$. Nonetheless, when Ω is locally quasiconvex and ρ is a Harnack metric-density, Lemma 2.8 below reveals that the identity map $\Omega \to \Omega_\rho$ is QC (and according to Proposition 2.9 even QS under the right circumstances). This is a good thing: we want Ω_ρ to be a quasiconformal deformation of Ω.

With this in mind, we pronounce the following definitions. First, we declare ρ to be a *bounded density* if the deformed space Ω_ρ is bounded, i.e., $\mathrm{diam}_\rho(\Omega) < \infty$.

Next, we call ρ a *Harnack density* provided it satisfies a uniform local Harnack type inequality: for all points x in Ω,

$$\frac{1}{H} \le \frac{\rho(y)}{\rho(x)} \le H \qquad \text{for all } y \in \lambda B(x). \tag{H}$$

Here $H = H_\rho \ge 1$ and $0 < \lambda < 1$ (generally λ will be small). Note that in contrast to the situation in [BKR98, p.637], the validity of (H) for some $0 < \lambda < 1$ need not mean a similar set of inequalities will hold for $\lambda = 1/2$. The condition (H) provides local control and permits the use of standard chaining type arguments; e.g., see Lemma 2.13.

We call ρ an *Ahlfors density* if the associated metric measure space $\Omega_\rho = (\Omega, d_\rho, \mu_\rho)$ is Ahlfors *upper* Q-regular (cf. §2.J); i.e., if μ_ρ satisfies a global upper

Ahlfors Q-regular volume growth estimate: there is a constant $A = A_\rho$ such that for all points x in Ω,

$$\text{(A)} \qquad \mu_\rho[B_\rho(x;r)] \le A\, r^Q \qquad \text{for all } r > 0.$$

The positive real number Q is generally the Hausdorff dimension of our space; it must agree with the number Q appearing in the definition of a Loewner space (a notion also discussed in §2.J). The volume growth condition (A) ensures that Ω_ρ satisfies an upper mass condition and so provides modulus estimates via Facts 2.15, 2.16, 2.17.

Below (in §2.H) we discuss the density $1/d$ which determines the quasihyperbolic distance; of course this is a continuous Harnack density, but in general $1/d$ does not satisfy the volume growth requirement (A).

We call ρ a *Koebe density* if Ω_ρ is non-complete and there is a constant $K = K_\rho$ such that $d_\rho(x) = \mathrm{dist}_\rho(x, \partial_\rho\Omega)$ enjoys the property

$$\text{(K)} \qquad \frac{1}{K} \le \frac{d_\rho(x)}{\rho(x)d(x)} \le K \qquad \text{for all } x \in \Omega.$$

(Note that when ρ is a Harnack density, $d_\rho(x) \ge (\lambda/H)\rho(x)d(x)$ always holds, and so it is the upper estimate which is needed.) For example, if $\rho = |f'|$ where f is a holomorphic homeomorphism defined in a subdomain Ω of the complex plane, then a classical theorem in univalent function theory asserts that ρ is a Koebe density with constant $K = 4$. As another example we note that Theorem E asserts that any Harnack Ahlfors density on a uniform Loewner space (with sufficiently 'thick' boundary) is a Koebe density; see [Her04, Theorem E]. We point out that when ρ is a Koebe density on (Ω, d), the identity map $(\Omega, k) \to (\Omega_\rho, k_\rho)$ is easily seen to be K_ρ-bilipschitz; here (Ω_ρ, k_ρ) denotes the quasihyperbolization of Ω_ρ.

We employ the terminology *conformal density* for a metric-density which is Harnack, Ahlfors, and Koebe. A basic example of a conformal density is $\rho = |f'|$ for any holomorphic homeomorphism $|f'|$ defined in a subdomain of the complex plane; we refer to [BKR98, Section 2] for other examples of conformal densities on the Euclidean unit ball. The reader should be aware that the phrase 'ρ is a conformal density' does not necessarily mean that the identity map $\Omega \to \Omega_\rho$ is quasiconformal (unless Ω is locally quasiconvex).

Here are some especially useful estimates which also provide information concerning the identity map $\Omega \to \Omega_\rho$ for certain densities. Roughly speaking, this map is locally bilipschitz (therefore quasiconformal) for Harnack densities and uniformly locally quasisymmetric for Harnack Koebe densities, provided Ω is locally quasiconvex. Proposition 2.9 gives a significant strengthening of this result.

2.8. Lemma. *Let $\rho : \Omega \to (0, \infty)$ be a Harnack density on a locally a-quasiconvex abstract domain (Ω, d). Put $\eta = \lambda/2a$. Then for all $z \in \Omega$,*

$$\frac{1}{H} \le \frac{d_\rho(x,y)}{\rho(z)|x-y|} \le aH \qquad \textit{for all points } x \ne y \textit{ in } \eta B(z);$$

in particular,

$$\frac{1}{H} \le \frac{\operatorname{diam}_\rho[\eta B(z)]}{\eta\rho(z)d(z)} \le 2aH.$$

If ρ is also a Koebe density, then for all $0 \le \vartheta \le \lambda/2C$ and all $x \in \Omega$,

$$C^{-1}\vartheta B(x) \subset \vartheta B_\rho(x) \subset C\vartheta B(x),$$

where $C = aHK$. Here $H = H_\rho$, $K = K_\rho$ and λ is the Whitney ball constant.

One immediate consequence of Lemma 2.8 is that the identity map $\Omega \to \Omega_\rho$ is metrically quasiconformal with linear dilatation aH^2. In addition, because of the definition of the associated measure (see (2.7)), a straightforward calculation reveals that this identity map is geometrically quasiconformal with inner dilatation H^Q and outer dilatation $(aH)^Q$. (Here we assume a Harnack density on a locally quasiconvex Ω.) It is therefore natural to inquire about possible quasisymmetry properties of this identity map.

Heinonen and Koskela proved that a quasiconformal map of bounded Ahlfors regular spaces, with domain a Loewner space and a linearly locally connected target space, is in fact quasisymmetric [HK98, Theorem 4.9]. The corollary to the following analog of their result is used in the proof of Theorem D; note that here our domain space is *not* assumed to be Ahlfors regular.

2.9. Proposition. *Let Ω be a bounded locally quasiconvex Q-Loewner space. Suppose ρ is a conformal density on Ω with Ω_ρ a bounded linearly locally connected space. Then the identity map $\Omega \to \Omega_\rho$ satisfies the weak-quasisymmetry condition*

$$\forall x, y, z \in \Omega: \quad |x-y| \le |x-z| \implies d_\rho(x,y) \le L d_\rho(x,z)$$

for some constant L which depends only on the data associated with Ω, ρ, Ω_ρ, and the ratios r, q given in the proof.

2.10. Corollary. *Let Ω be a bounded quasiconvex Q-Loewner space. Suppose ρ is a conformal density on Ω and Ω_ρ is a bounded Q-Loewner space. Then the identity map $\Omega \to \Omega_\rho$ is quasisymmetric with a distortion function which depends only on the data associated with Ω, ρ, Ω_ρ, and the ratios r, q given in the proof of Proposition 2.9.*

As an exercise to help understand the various properties of these metric-densities, the interested reader can provide a proof for the following [Her06, Lemma 2.6].

2.11. Lemma. *Suppose (Ω, d, μ) is a locally a-quasiconvex abstract domain. Let τ be a positive Borel function on Ω which is locally bounded away from 0 and ∞. Put $\Delta = \Omega_\tau$. If σ is a metric-density on Δ, then its pull-back $\rho = \sigma\tau$ is a metric-density on Ω, $\Omega_\rho = \Delta_\sigma$, and*

(a) *ρ and σ either are, or are not, both Ahlfors regular (with $A_\rho = A_\sigma$),*

(b1) *if σ, τ are both Koebe, then so is ρ with $K_\rho = K_\sigma K_\tau$,* (c1) *if* σ, τ *are both Harnack, then so is* ρ *with* $H_\rho = H_\sigma H_\tau$.

On the other hand, if ρ *is a metric-density on* Ω, *then its push-forward* $\sigma = \rho\tau^{-1}$ *is a metric-density on* Δ, $\Delta_\sigma = \Omega_\rho$, (a) *holds, and*

(b2) *if* ρ, τ *are Koebe, then so is* σ *with* $K_\sigma = K_\rho K_\tau$,

(c2) *if* ρ, τ *are Harnack and* τ *is Koebe, then* σ *is Harnack with* $H_\sigma = H_\rho H_\tau$.

2.H. Quasihyperbolic Distance and Geodesics. The quasihyperbolic distance in an abstract domain (Ω, d) is defined by

$$k(x,y) = k_\Omega(x,y) := \inf \ell_k(\gamma) = \inf \int_\gamma \frac{ds}{d(z)}$$

where the infimum is taken over all rectifiable curves γ which join x, y in Ω. The *quasihyperbolization* of an abstract domain (Ω, d) is the metric space (Ω, k) obtained by using quasihyperbolic distance. It is not hard to see that (Ω, k) is complete, provided the identity map $(\Omega, \ell) \to (\Omega, d)$ is a homeomorphism; see [BHK01, Proposition 2.8]. Thus by the Hopf-Rinow theorem ([Gro99, p.9], [BBI01, p.51], [BH99, p.35]), every locally compact abstract domain has a proper (hence geodesic) quasihyperbolization.

We call the geodesics in (Ω, k) *quasihyperbolic geodesics*; see §2.D. Note that when ρ is a Koebe density on Ω, the identity map $(\Omega, k) \to (\Omega_\rho, k_\rho)$ is bilipschitz and we find that quasihyperbolic geodesics in Ω are quasihyperbolic quasigeodesics in Ω_ρ; that is, a geodesic in (Ω, k) will be a quasigeodesic in (Ω_ρ, k_ρ) (the quasihyperbolization of Ω_ρ).

We remind the reader of the following basic estimates for quasihyperbolic distance, first established by Gehring and Palka [GP76, Lemma 2.1]:

$$k(x,y) \geq \log\left(1 + \frac{\ell(x,y)}{d(x) \wedge d(y)}\right) \geq j(x,y) = \log\left(1 + \frac{|x-y|}{d(x) \wedge d(y)}\right) \geq \left|\log \frac{d(x)}{d(y)}\right|.$$

See also [BHK01, (2.3),(2.4)]. The first inequality above is a special case of the more general (and easily proved) inequality,

$$\ell_k(\gamma) \geq \log\left(1 + \frac{\ell(\gamma)}{d(x) \wedge d(y)}\right)$$

which holds for any rectifiable curve γ with endpoints x, y.

An immediate consequence of the above inequalities is that the identity map $(\Omega, k) \to (\Omega, d)$ is continuous; indeed,

$$B_k(x;R) \subset (e^R - 1)B(x) \qquad \text{for all } x \in \Omega \text{ and all } R > 0,$$

where $B_k(x;R)$ denotes the R-ball centered at x in (Ω, k). It is important to know when this map will be a homeomorphism (which, according to [BHK01, Lemma A.4, p.92], will be the case if and only if the identity map $(\Omega, \ell) \to (\Omega, d)$ is a homeomorphism). The following provides quantitative information concerning this question; it is easy to verify via simple estimates for the quasihyperbolic lengths of the 'promised short arcs'.

2.12. Lemma. *Suppose that (Ω, d) is a locally a-quasiconvex abstract domain. Then for all $x \in \Omega$ and all $R > 0$,*

$$\tau B(x) \subset B_k(x; R) \qquad \textit{provided } 0 \le \tau \le \min\{\lambda, R/[a(1+R)]\}.$$

As an exercise, the reader can check that for a domain in $\mathbb{R}^n$, $|x-y| \le [d(x)+d(y)]/2 \implies k(x,y) \le 2$. Thus $(2/3)B(x) \subset B_k(x; 2)$.

The Harnack inequality (H), as stated in §2.G, only requires that ρ be essentially constant on Whitney type balls. We can do the usual chaining type arguments to see that such a density ρ will satisfy a Harnack type inequality on much bigger sets, of course with a change in the Harnack constant. Here is a useful example of this phenomena.

2.13. Lemma. *Let ρ be a Harnack density on an abstract domain (Ω, d). If $x, y \in \Omega$ satisfy $k(x,y) \le K$, then $1/H_1 \le \rho(y)/\rho(x) \le H_1$, where $H_1 = H_1(K, H_\rho, \lambda)$.*

We conclude this subsection with a covering lemma for quasihyperbolic geodesics.

2.14. Lemma. *Suppose that (Ω, d) is a locally a-quasiconvex abstract domain. Let γ be a quasihyperbolic geodesic segment or ray in Ω with endpoint x_0. Let $x_0, x_1, x_2, \ldots$ be successive points along γ with $k(x_i, x_{i-1}) = K \le \log(1+\tau)$ where $\tau = \min\{\lambda, 1/2a\}$. Then the balls $B_i = \tau B(x_i)$ cover γ and have bounded overlap: $\sum \chi_{B_i} \le C\chi_{\cup B_i}$, where $C = 1 + 4/K$.*

2.I. Modulus and Capacity. For $p \ge 1$ we define the *p-modulus* of a family Γ of curves in a metric measure space (X, d, μ) by

$$\operatorname{mod}_p \Gamma := \inf \int \rho^p \, d\mu,$$

where the infimum is taken over all Borel functions $\rho : X \to [0, \infty]$ satisfying $\int_\gamma \rho \, ds \ge 1$ for all locally rectifiable curves $\gamma \in \Gamma$. Then the *p-modulus* of a pair of disjoint compact sets $E, F \subset X$ is

$$\operatorname{mod}_p(E, F; X) := \operatorname{mod}_p \Gamma(E, F; X)$$

where $\Gamma(E, F; X)$ is the family of all curves joining the sets E, F in X. We also let $\Gamma_{\mathrm{r}}(E, F; X)$ be the subfamily of $\Gamma(E, F; X)$ consisting of the rectifiable paths joining E, F.

An important property is that under fairly general circumstances, $\operatorname{mod}_p(E, F; X)$ agrees with the *p-capacity* of the pair E, F. There is extensive literature regarding these "capacity equals modulus" results; for a start, see [HK98, Proposition 2.17].

For the reader's convenience, we cite the following modulus estimates. First we have the standard Long Curves Estimate; see [HK98, 3.15].

2.15. Fact. *Let $x \in X$ and suppose that the upper mass condition $\mu[B(x; R)] \le MR^p$ holds for some $R > 0$. Let Γ be a family of curves in $B(x; R)$ and suppose that each $\gamma \in \Gamma$ has arclength $\ell(\gamma) \ge L > 0$. Then*

$$\operatorname{mod}_p \Gamma \le L^{-p} \mu[B(x; R)] \le M(R/L)^p.$$

Next we record the Spherical Ring Estimate; see [HK98, 3.14, p.17].

2.16. Fact. *Let $x \in X$, $0 < 2r \le R$, and suppose that the upper mass condition $\mu[B(x;t)] \le Mt^p$ holds for all $0 < t < r + R$. Then*

$$\operatorname{mod}_p(\bar{B}(x;r), X \setminus B(x;R); X) \le C\,(\log(R/r))^{1-p}\,,$$

where $C = 2^{p+1}M/\log 2$.

Finally, we require the following Basic Modulus Estimate; see [BKR98, Lemma 3.2].

2.17. Fact. *Let (X, d, μ) be a metric measure space. Assume that ρ is a metric-density on X whose associated measure (2.7) satisfies the Ahlfors volume growth condition (A) at some point $x \in E \subset X$. Suppose that $L > \lambda \ge \operatorname{diam}_\rho E$, and that Γ is some family of curves γ in X each having one endpoint in E and satisfying $\ell_\rho(\gamma) \ge L$. Then*

$$\operatorname{mod}_Q \Gamma \le C\,(\log\,(1 + L/\lambda))^{1-Q}\,,$$

where $C = 2^{Q+1}A/\log 2$.

2.18. Corollary. *Let (X, d, μ) be a metric measure space. Assume that for some point $x \in E \subset X$, the upper mass condition $\mu[B(x;r)] \le Mr^Q$ holds for all $r > 0$. Suppose that Γ is a family of curves γ in X each having one endpoint in E and satisfying $\ell(\gamma) \ge L > \operatorname{diam} E$. Then*

$$\operatorname{mod}_Q \Gamma \le C\,(\log\,(1 + L/\operatorname{diam} E))^{1-Q}\,,$$

where $C = 2^{Q+1}M/\log 2$.

In Euclidean space $\mathbb{R}^n$, the n-modulus is also called the *conformal modulus* and simply denoted by $\operatorname{mod}(\cdot)$. Below we state some well-known geometric estimates for the conformal modulus $\operatorname{mod}(E, F; \Omega)$. Here and elsewhere in these notes,

$$\Delta(E, F) := \operatorname{dist}(E, F)/\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}$$

is the *relative distance* between the pair E, F of nondegenerate disjoint continua.

2.19. Facts. *Let E, F be disjoint compact sets in $\mathbb{R}^n$.*

(a) *If E, F are separated by the spherical ring $B(x;s) \setminus \bar{B}(x;t)$, then*
$$\operatorname{mod}(E, F; \mathbb{R}^n) \le \omega_{n-1}\,(\log(s/t))^{1-n}\,.$$

(b) *If $E \cap S(x;r) \ne \emptyset \ne F \cap S(x;r)$ for all $t < r < s$, then*
$$\operatorname{mod}(E, F) \ge \sigma_n \log(s/t).$$

(c) *If both E and F are connected, then*
$$\sigma_n \log(1 + 1/\Delta(E, F)) \le \operatorname{mod}(E, F; \mathbb{R}^n) \le \Omega_n(1 + 1/\Delta(E, F))^n.$$

(d) (Comparison Principle) *If $A, B, E, F \subset \Omega$ with A, B also compacta, then*
$$\operatorname{mod}(E, F; \Omega) \ge 3^{-n} \min\{\operatorname{mod}(E, A; \Omega), \operatorname{mod}(F, B; \Omega), I\},$$
where $I = \inf\{\operatorname{mod}(\alpha, \beta; \Omega) \mid \alpha \in \Gamma_{\mathrm{r}}(E, A; \Omega), \beta \in \Gamma_{\mathrm{r}}(F, B; \Omega)\}$.

(e) (Teichmüller Estimate) *If E, F are both connected, then for all $x, y \in E$ and $z, w \in F$*

$$\operatorname{mod}(E, F; \mathbb{R}^n) \geq \tau\left(\frac{|x-z||y-w|}{|x-y||z-w|}\right)$$

where $\tau(r)$ is the capacity of the Teichmüller ring

$$\mathbb{R}^n \setminus \{-1 \leq x_1 \leq 0 \;\; or \;\; x_1 \geq r\};$$

i.e, $\tau(r) = \operatorname{mod}([-e_1, 0], [re_1, \infty]; \mathbb{R}^n)$.

(f) *There exists $\lambda = \lambda(n) \in [6, 5e^{(n-1)/2})$ such that when E, F are both connected and $\Delta(E, F) \geq 1$,*

$$2^{1-n}\omega_{n-1}[\log(\lambda\Delta(E,F))]^{1-n} \leq \operatorname{mod}(E, F; \mathbb{R}^n) \leq \omega_{n-1}[\log(\Delta(E,F))]^{1-n}.$$

(g) (Carleman Inequality) *For $E \subset \Omega$,*

$$\operatorname{mod}(E, \partial\Omega; \Omega) \geq n^{n-1}\omega_{n-1}\left(\log(|\Omega|/|E|)\right)^{1-n}.$$

Here σ_n and ω_{n-1}, Ω_n are the spherical cap constant and the measures of the $(n-1)$-sphere, n-ball respectively.

Most of these estimates can be found in [Väi71] or [Vuo88]. Lemma 2.5 in [BH06] gives a precise formula for $\operatorname{mod}(E, F; \mathbb{R}^n)$ in the case when E, F are disjoint closed balls.

2.J. Ahlfors Regular and Loewner Spaces. A metric measure space (X, d, μ) is *Ahlfors Q-regular* provided there exists a finite constant $M = M_\mu$ such that for all $x \in X$ and all $0 < r \leq \operatorname{diam}\Omega$,

$$M^{-1}r^Q \leq \mu[B(x;r)] \leq M\, r^Q.$$

The positive real number Q will then be the Hausdorff dimension of (X, d), and the Q-dimensional Hausdorff measure $\mathcal{H}^Q$ on X will also satisfy the above inequalities (possibly with a change in the constant M). A metric space (X, d) is Ahlfors Q-regular if $(X, d, \mathcal{H}^Q)$ is Ahlfors Q-regular. We use the adjectives *upper* or *lower* to indicate that only one of these inequalities is in force, and—in the abstract domain setting—add the adjective *locally* to mean that the required inequality holds (or, inequalities hold) for Whitney balls (i.e., for radii $0 < r \leq \lambda d(x)$).

There is an interesting result which gives upper estimates for the Assouad dimension of subsets of Ahlfors regular spaces. See [BHR01, 3.12], [DS97, 5.8], [Luu98, 5.2].

2.20. Fact. *Suppose X is an Ahlfors Q-regular space and let $M \subset X$. Then $\dim_{\mathcal{A}} M < Q$ if and only if M is porous in X; the constants depend only on each other and the $\mathcal{H}^Q$-regularity constant.*

The notion of a Loewner space was introduced by Heinonen and Koskela in their study [HK98] of quasiconformal mappings of metric spaces; Heinonen's recent monograph [Hei01] renders an enlightening account of these ideas. A path-connected metric measure space (X, d, μ) is a *Q-Loewner space*, $Q > 1$, provided the *Loewner control function*

$$\varphi(t) := \inf\{\mathrm{mod}_Q(E, F; X) : \Delta(E, F) \le t\}$$

is strictly positive for all $t > 0$; here E, F are non-degenerate disjoint continua in Ω and

$$\Delta(E, F) := \mathrm{dist}(E, F)/\min\{\mathrm{diam}(E), \mathrm{diam}(F)\}$$

is the relative size of the pair E, F. Note that we always have $Q \ge \dim_{\mathcal{H}}(\Omega) \ge 1$.

When (Ω, d, μ) is an n-Loewner space with $\Omega \subset \mathbb{R}^n$ a domain and d, μ are Euclidean distance and Lebesgue n-measure respectively, we simply call Ω a *Loewner domain*. This is a generalization of Väisälä's notion of a *broad domain* (which he introduced in his analysis [Väi89, 2.15] of space domains QC equivalent to a ball, and also used in his study [NV91, 3.8] of John disks), which in turn is an analog of the *quasiextremal distance domains* first studied by Gehring and Martio [GM85].

We call $\Omega \subsetneq \mathbb{R}^n$ a *ψ-QED domain* if $\psi : [0, \infty) \to [0, \infty)$ is a homeomorphism and for all disjoint continua E, F in Ω,

$$\mathrm{mod}(E, F; \Omega) \ge \psi(\mathrm{mod}(E, F; \mathbb{R}^n)).$$

Clearly, $\psi(t) \le t$ is a necessary restriction on such ψ. Also, every ψ-QED domain is Loewner. The typical nonlinear functions ψ that arise in the literature have the form $\psi_{p,M}(t) = M^{-1}\min\{t^p, t^{1/p}\}$ with $p, M \ge 1$, a condition we call M-QED^p, or simply M-QED when $p = 1$.

The most important, and original, inequalities of this form are the *M-QED conditions* corresponding to $\psi(t) = t/M$ for some constant $M \ge 1$. This idea was introduced by Gehring and Martio who called such regions *quasiextremal distance domains*. The terminology arises from the fact that the quantity $\mathrm{mod}(E, F; \Omega)^{1/(1-n)}$ is the *extremal distance* between E and F in Ω. When we speak of a *QED domain* or a *QED condition*, we always mean an M-QED domain or an M-QED condition for some $M \ge 1$.

As in [HK96] we can consider the location of the continua E, F as well as looking at special types of continua. In particular we can relax the ψ-QED inequality by requiring it to hold only for all disjoint closed balls (or just closed Whitney balls) to get the class ψ-QED$_\mathrm{b}$ (or ψ-QED$_\mathrm{wb}$, respectively). Precise definitions can be found in [BH06].

Every a-uniform domain in $\mathbb{R}^n$ is M-QED for some $M = M(a, n)$; this follows easily from Jones' extension result for Sobolev spaces [Jon81, Theorem 1]. Also, it is trivially true that

$$\mathrm{QED} \implies \psi-\mathrm{QED} \implies \psi-\mathrm{QED_b} \implies \psi-\mathrm{QED_{wb}}.$$

The converse of the middle implication fails; see[HK96, Example 4.1] and [BH06, Example 4.2]. According to [BH06, Theorem 3.3], the last implication is reversible modulo a quantitative change in ψ. In addition, we always have

$$\psi-\mathrm{QED} \iff \mathrm{Loewner} \implies \mathrm{QED_{wb}} \implies \psi-\mathrm{QED_{wb}} \iff \mathrm{Loewner_{wb}}$$

where the last condition means that the Loewner condition is assumed only for Whitney balls. Examples 4.2 and 4.3 in [BH06] illustrate that in general the converses of the middle two implications fail to hold. The first equivalence is established in [BH06, Theorem 1.3]. It remains open as to whether or not Loewner domains (i.e. ψ-QED domains) are always QED.

2.K. Slice Conditions. There are various so-called *slice conditions* each designed to handle their own specific problem. The ideas here are due to Buckley et al. and his exposition [Buc03] is the place to begin reading about this topic. He and his many co-authors have utilized an assortment of slice conditions to investigate all kinds of different problems.

A non-empty bounded open set $S \subset X$ is called a *C-slice separating x, y* provided

$$\forall\, \alpha \in \Gamma(x,y): \quad \ell(\alpha \cap S) \geq \operatorname{diam}(S)/C$$

and

$$C^{-1}B(x) \cap S = \emptyset = S \cap C^{-1}B(y)\,.$$

A *set of C-slices for $x, y \in X$* is a collection $\mathcal{S}$ of pairwise disjoint C-slices separating x, y in X. One can show (see [BS03, (2.1)]) that the cardinality of any such set $\mathcal{S}$ of C-slices separating x, y is always bounded by $\#\mathcal{S} \leq C^2 k(x,y)$. We are interested in knowing when we can reverse this inequality. Since there may be no C-slices separating x, y, we consider the quantity

$$d_{\mathrm{ws}}(x,y) = d_{\mathrm{ws}}(x,y;C) = d^X_{\mathrm{ws}}(x,y;C) := 1 + \sup \#\mathcal{S}$$

where the supremum is taken over all $\mathcal{S}$ which are sets of C-slices in X separating x, y, and $\#\mathcal{S}$ denotes the cardinality of $\mathcal{S}$.

We call (X,d) a *weak C-slice space* provided for all $x, y \in X$,

$$k(x,y) \leq C\, d_{\mathrm{ws}}(x,y;C),$$

Thus in these spaces $d_{\mathrm{ws}}(x,y) \simeq k(x,y)$, at least when $k(x,y) \geq 2$. The weak slice condition was introduced in [BO99, Section 5]; see also [BS03], [Buc03], [Buc04]. When the weak C-slice space (X,d) is a domain $\Omega \subsetneq \mathbb{R}^n$, we call Ω a weak C-slice domain.

The following rather technical lemma is quite useful for obtaining an upper bound for the cardinality of a set of slices; in weak slice spaces it provides an upper bound for quasihyperbolic distances. It is the case $\alpha = 0$ of [BS03, Lemma 2.17].

2.21. Lemma. *Let Γ be a 1-rectifiable subset of a rectifiably connected metric space (X,d). Suppose $\varphi : \Gamma \to [\varepsilon, \infty)$ (with $\varepsilon > 0$) and $\mathcal{S}$ is a collection of disjoint non-empty bounded subsets of X. Suppose also that there exist positive constants b, c such that*

(a) $\forall S \in \mathcal{S}: \quad \ell(S \cap \Gamma) \geq c \operatorname{diam}(S)$,
(b) $\forall S \in \mathcal{S}\,,\ \forall z \in S \cap \Gamma: \quad \varphi(z) \leq \operatorname{diam}(S)$,

(c) $\forall t > 0 : \quad \ell(\varphi^{-1}(0,t]) \leq b\,t\,.$

Then the cardinality of $\mathcal{S}$ is at most $\#\mathcal{S} \leq 2(b/c)\log_2(4\ell(\Gamma)/c\varepsilon)$.

3. Uniform Spaces

Roughly speaking, a space is uniform provided points in it can be joined by so-called bounded turning twisted double cone arcs, i.e. paths which are not too long and which stay away from the regions boundary. Uniform domains in Euclidean space were first studied by John [Joh61] and Martio and Sarvas [MS79] who proved injectivity and approximation results for them. They are well recognized as being the 'nice' domains for quasiconformal function theory as well as many other areas of geometric analysis (e.g., potential theory); see [Geh87] and [Väi88]. Every (bounded) Lipschitz domain is uniform, but generic uniform domains may very well have fractal boundary. Recently, uniform subdomains of the Heisenberg groups, as well as more general Carnot groups, have become a focus of study; see [CT95], [CGN00], [Gre01].

3.A. Euclidean Setting. When our uniform space (see the definition given below in §3.B) (Ω, d) is a domain $\Omega \subset \mathbb{R}^n$ with Euclidean distance, we simply call Ω a *uniform domain*. Every plane uniform domain is a quasicircle domain (each of its boundary components is either a point or a quasicircle), and a finitely connected plane domain is uniform if and only if it is a quasicircle domain. However, the plane punctured at the integers is not uniform. Such nice topological information is not true for uniform domains in higher dimensions. For example, a ball with a radius removed is uniform; this is not true when $n = 2$.

For domains in $\mathbb{R}^n$ we can consider uniformity both with respect to the Euclidean distance and with respect to the induced length metric also. The latter class of domains are usually called *inner uniform*; cf. [Väi98]. For example, a slit disk in the plane is not uniform (with respect to Euclidean distance) but it is an inner uniform domain. On the other hand, an infinite strip, or the inside of an infinite cylinder in space, is not uniform nor inner uniform. The region between two parallel planes is not uniform nor inner uniform. Every quasiball is uniform.

3.B. Measure Metric Space Setting. Following [BHK01], a *uniform space* is an abstract domain (so, a non-complete, locally complete, rectifiably connected metric space) (Ω, d) with the property that there is some constant $a \geq 1$ such that each pair of points can be joined by an a-uniform arc. A rectifiable arc γ joining x, y in Ω is an *a-uniform arc* provided

$$\ell(\gamma) \leq a|x - y|$$

and

$$\min\{\ell(\gamma[x,z]), \ell(\gamma[y,z])\} \leq a\,d(z) \qquad \text{for all } z \in \gamma.$$

Here $\ell(\gamma)$ is the arclength of γ and $\gamma[x,z]$ denotes the subarc of γ between x, z. The second inequality above ensures that Ω contains the twisted double cone $\cup\{B(z;\ell(z)/a) : z \in \gamma\}$ where $\ell(z)$ denotes the left-hand-side of this inequality;

the first inequality asserts that this twisted double cone is not too 'crooked'. Consequently, we call γ a *double a-cone arc* if it satisfies the second inequality above (the phrases *cigar arc* and *corkscrew* are also used).

3.C. Basic Information. An important, and characteristic, property of uniform spaces is that quasihyperbolic geodesics are uniform arcs. (See [GO79, Theorems 1,2] for domains in Euclidean space and [BHK01, Theorem 2.10] for general metric spaces.) Slight alterations to the proof of [BHK01, Theorem 2.10] yield the following generalization of this property.

3.1. Fact. *In an a-uniform space, quasihyperbolic c-quasigeodesics are b-uniform arcs where $b = b(a,c)$.*

In general, quasihyperbolic geodesics may not exist; see [Väi99, 3.5] for an example due to P. Alestalo. However, one can still show that quasihyperbolically short arcs are uniform arcs. One can prove that boundary points in a locally compact uniform space can be joined by quasihyperbolic geodesics, and these geodesics are still uniform arcs.

Another crucial piece of information is a characterization of uniformity due to Gehring and Osgood [GO79, Theorems 1,2]; Bonk, Heinonen, and Koskela [BHK01, Lemma 2.13] verified the necessity of this condition for the metric space setting, while the Gehring-Osgood argument can be modified to establish the sufficiency. Recalling the basic estimates for quasihyperbolic distance, we see that uniform spaces are precisely those abstract domains in which the quasihyperbolic distance is bilipschitz equivalent to the j distance. See also Theorem 6.1.

3.2. Fact. *An abstract domain is a-uniform if and only if $k(x,y) \le b\, j(x,y)$ for all points x, y. The constants a and b depend only on each other.*

We conclude this subsection with a useful fact regarding bounded uniform spaces; see [Her06, Lemmas 2.12,2.13].

3.3. Lemma. *Let ρ be a Harnack Koebe density on a bounded a-uniform space (Ω, d). Suppose that Ω_ρ is bounded and that quasihyperbolic geodesics in Ω are double a-cone arcs in Ω_ρ. Then for any positive constant C,*

$$\operatorname{diam}_\rho[CB(z)] \simeq d_\rho(z) \qquad \text{for all } z \in \Omega,$$

where the constant depends only on $C, H_\rho, K_\rho, a, \lambda$ and the quantity q given in the proof.

4. Gromov Hyperbolicity

Good sources for information concerning Gromov hyperbolicity include [BHK01], [BBI01], [BS00], [BH99], [Bon96] and especially the references mentioned in these works. Väisälä has an especially nice treatment [Väi05a] of Gromov hyperbolicity for spaces which are not assumed to be geodesic nor proper. Note however that Bonk and Schramm have demonstrated that every Gromov δ-hyperbolic metric space can be isometrically embedded into some complete geodesic δ-hyperbolic space; see [BS00, Theorem 4.1].

Hästö [Häs06] has an intriguing result giving a striking contrast between the hyperbolicity of (Ω, j) versus that of $(\Omega, \tilde{j})$ for $\Omega \subsetneq \mathbb{R}^n$: the latter space is always Gromov hyperbolic whereas the former is Gromov hyperbolic precisely when Ω has exactly one boundary point. This is quite surprising as these spaces are bilipschitz equivalent (indeed, $\tilde{j} \le j \le 2\tilde{j}$). It is known that for intrinsic spaces, so also for geodesic spaces, Gromov hyperbolicity is preserved under (L, C)-quasiisometries. In particular, Hästö's result illustrates the failure of this property in the non-intrinsic setting.

4.A. Thin Triangles Definition. A geodesic metric space is *Gromov hyperbolic* if its geodesic triangles are δ*-thin* for some $\delta > 0$, which means that each point on the edge of any geodesic triangle is within distance δ of some point on one of the other two edges. That is, if $[x,y] \cup [y,z] \cup [z,x]$ is a geodesic triangle, then for all $u \in [x,y]$, $\mathrm{dist}(u, [x,z] \cup [y,z]) \le \delta$. (Recall that $[x,y]$ denotes some arbitrary, but fixed, geodesic joining x, y.)

There is a more general definition which applies to non-geodesic spaces. It is based on the *Gromov product*

$$(x|y)_w := \frac{1}{2}(|x-w| + |y-w| - |x-y|) \qquad \text{for points } x, y, w \text{ in the space}.$$

The Gromov product is useful even in geodesic spaces; it can be extended to the Gromov boundary and then used to define a canonical conformal gauge there.

Roughly speaking, all simply connected manifolds with negative curvature are Gromov hyperbolic; e.g., every CAT(κ) space with $\kappa < 0$. For a specific example, consider any bounded strictly pseudoconvex domain Ω (with sufficiently smooth boundary) in complex n-space together with any of the classical hyperbolic distances h; a result of Balogh and Bonk [BB00] asserts that (Ω, h) is a Gromov hyperbolic space (with $\partial_G \Omega = \partial \Omega$, the Euclidean boundary, and canonical conformal gauge determined by the Carnot-Carathéodory distance on $\partial\Omega$).

4.B. Gromov Boundary. The *Gromov boundary* $\partial_G H$ of a proper geodesic Gromov hyperbolic metric space (H, h) is defined as the set of equivalence classes of geodesic rays, with two such rays being equivalent if they have finite Hausdorff distance. That is, $\partial_G H$ is the geodesic boundary of H; see §2.D. An alternative description can be given in terms of (equivalent) sequences which *converge at infinity*; in particular, this allows us to extend the Gromov product to the boundary (cf. [Väi05a, 5.7] or [BH99, pp. 431-436]). This in turn yields a canonically defined conformal gauge on the Gromov boundary generated by the quasimetrics

$$q_{w,\varepsilon}(\xi, \eta) = \exp[-\varepsilon(\xi|\eta)_w] \qquad \text{for points } \xi, \eta \in \partial_G H.$$

For $0 < \varepsilon < \varepsilon(\delta) = \log 2/(4\delta)$, each quasimetric q_ε is bilipschitz equivalent to an honest metric on the boundary, and all these metric spaces are QS equivalent to each other; in particular they all generate the same topology on $\partial_G H$, and $\partial_G H$ is compact. See Fact 2.1, [BH99, Proposition 3.21, pp.435-436], [BHK01, p.18].

4.C. Connection with Uniform Spaces. Bonk, Heinonen and Koskela established the following fundamental connection between uniform spaces and Gromov hyperbolicity; see [BHK01, Proposition 2.8, Theorem 3.6]

4.1. Fact. *The quasihyperbolization (Ω, k) of a locally compact a-uniform space (Ω, d) is proper, geodesic and δ-hyperbolic where $\delta = \delta(a) = 10000a^8$. When (Ω, d) is bounded, (Ω, k) is roughly κ-starlike with $\kappa = 5000a^8$.*

In fact they also prove that the Gromov boundary $\partial_G\Omega$ 'is' the boundary $\dot{\partial}\Omega$ of Ω in the one-point extension $\dot{\Omega}$ of Ω; see [BHK01, Proposition 3.12]. Moreover, in the bounded case, the canonical gauge on $\partial_G\Omega$ is naturally quasisymmetrically equivalent to the conformal gauge determined by d on $\partial\Omega$. See [Väi05b] for similar results in the Banach space setting.

5. Uniformization

The celebrated Riemann Mapping Theorem asserts that every simply connected proper subdomain of the plane can be mapped conformally onto the unit disk, and hence supports a bounded conformal uniformizing metric-density, namely, $\rho = |f'|$ where f is the Riemann map. Koebe proved a similar result for finitely connected plane domains: any one of these can always be conformally mapped onto a circle domain (meaning that each boundary component is either a point or a circle).

In space, every conformal map is (the restriction of) a Möbius transformation, and thus the only space regions conformally equivalent to a ball are balls and half-spaces. The problem of determining which space domains are QC equivalent to a ball has been investigated for more than four decades by now (see [GV65]), and the most significant result (that I know of) is Väisälä's characterization in [Väi89] describing the cylindrical domains ($\Omega = D \times \mathbb{R} \subset \mathbb{R}^3$) which are QC equivalent to $\mathbb{B}^3$.

5.A. Uniformization Problem. Here we consider the metric space analog of the Riemann Mapping Problem. We seek to characterize the abstract domains which can be quasiconformally deformed into a uniform space. We ask the question: What are necessary and sufficient conditions for an abstract domain to support a conformal uniformizing metric-density? Theorem A provides an initial answer.

5.B. BHK Uniformization. Bonk, Heinonen and Koskela developed a uniformization theory, which they call *dampening*, valid for proper geodesic Gromov hyperbolic spaces. Their theory produces the following; see [BHK01, Proposition 4.5, Chapter 5]. (They established far more than we mention here:-)

5.1. Fact. *Let (H, h) be an unbounded proper geodesic Gromov δ-hyperbolic space. Fix a base point $w \in H$. For $\varepsilon > 0$ define $\rho_\varepsilon(x) = \exp[-\varepsilon h(x, w)]$ and let $H_\varepsilon = (H, d_\varepsilon)$ where $d_\varepsilon = h_{\rho_\varepsilon}$. Then:*

(a) *The geodesics in* H *are double* a*-cone arcs in* H_ε *with* $a = a(\varepsilon, \delta) = e^{1+8\varepsilon\delta}$.
(b) *There is a constant* $\varepsilon_0 = \varepsilon_0(\delta)$ *such that for all* $\varepsilon \in (0, \varepsilon_0]$,

$$\forall x, y \in H\,, \forall \text{ geodesics } [x,y]: \quad \ell_\varepsilon[x,y] \le 20\, d_\varepsilon(x,y);$$

here $\ell_\varepsilon = \ell_{\rho_\varepsilon}$.

In fact, H_ε is always bounded, and thus when $\varepsilon \le \varepsilon_0$ we see that (H, h) has been deformed, or dampened, (via the natural metric-density ρ_ε) to a bounded 20-uniform space H_ε.

We briefly describe their theory in the special case which concerns us.

We consider a locally compact abstract domain (Ω, d) with the property that the identity map $(\Omega, \ell) \to (\Omega, d)$ is a homeomorphism (so the identity $(\Omega, k) \to (\Omega, d)$ is also a homeomorphism) and such that its quasihyperbolization (Ω, k) is a Gromov hyperbolic space. According to Fact 5.1, the space (Ω, k) admits a uniformizing density of the form

$$\rho_\varepsilon(x) := \exp[-\varepsilon k(x, w)];$$

here $w \in \Omega$ is a fixed base point and $\varepsilon > 0$ a sufficiently small parameter. More precisely, when (Ω, k) is δ-hyperbolic and $0 < \varepsilon < \varepsilon(\delta)$, the quasihyperbolic geodesics in Ω are 20-uniform arcs in (Ω, d_ε). (A careful check of BHK shows that $\varepsilon(\delta) = [42(5 + 192\delta + 1920\delta^2)]^{-1} \le (300 \max\{1, \delta\})^{-2}$. :-) Here d_ε stands for the distance function obtained by conformally deforming k via the metric-density ρ_ε. Since k was obtained from the original distance function d via the quasihyperbolic density $1/d$, $\Omega_\varepsilon = (\Omega, d_\varepsilon)$ is a conformal deformation of (Ω, d) via the metric-density

$$\pi_\varepsilon(x) := \rho_\varepsilon(x)/d(x) = d(x)^{-1} \exp(-\varepsilon k(x, w))\,;$$

again, π_ε will be a uniformizing density when $0 < \varepsilon < \varepsilon(\delta) = (300 \max\{1, \delta\})^{-2}$.

In order to determine when π_ε will be a Harnack or Koebe density, we need the following information concerning ρ_ε (see [BHK01, (4.4),(4.6),(4.17)]):

$$e^{-\varepsilon k(x,y)} \le \frac{\rho_\varepsilon(x)}{\rho_\varepsilon(y)} \le e^{\varepsilon k(x,y)}, \tag{5.2}$$

$$\frac{\rho_\varepsilon(x)}{e\varepsilon} \le d_\varepsilon(x) \le (2e^{\varepsilon\kappa} - 1)\frac{\rho_\varepsilon(x)}{\varepsilon}. \tag{5.3}$$

The first set of inequalities (5.2) hold for all points $x, y \in \Omega$ and all $\varepsilon > 0$. They guarantee that the identity map $(\Omega, k) \to (\Omega, d_\varepsilon)$ is locally bilipschitz, so (Ω, d_ε) is locally compact and rectifiably connected. On the other hand, $\Omega_\varepsilon = (\Omega, d_\varepsilon)$ is non-complete, so we can form $\bar{\Omega}_\varepsilon$, put $\partial_\varepsilon\Omega = \bar{\Omega}_\varepsilon \setminus \Omega$ and let $d_\varepsilon(x) = \operatorname{dist}_\varepsilon(x, \partial_\varepsilon\Omega)$. (See [BHK01, pp.27-28]). We find that the leftmost inequality in (5.3) holds for all $x \in \Omega$ and any $\varepsilon > 0$. However, in order to obtain the rightmost inequality in (5.3), we must further require that (Ω, k) be roughly κ-starlike.

Now using (5.2), and obvious inequalities for $1/d$, we deduce that

$$e^{-2R} \le e^{-(\varepsilon+1)R} \le \frac{\pi_\varepsilon(y)}{\pi_\varepsilon(x)} \le e^{(\varepsilon+1)R} \le e^{2R}$$

for all points x, y with $k(x,y) \le R$. According to Lemma 2.12, when (Ω, d) is locally a-quasiconvex we have

$$\tau B(x) \subset B_k(x;R) \qquad \text{provided } 0 < \tau \le \min\{\lambda, R/[a(1+R)]\}.$$

Taking $R = a$ (say) and $\tau = \min\{\lambda, 1/2a\}$ we find that for all $x \in \Omega$ and all $y \in \tau B(x)$,

$$e^{-2a} \le e^{-(\varepsilon+1)a} \le \frac{\pi_\varepsilon(y)}{\pi_\varepsilon(x)} \le e^{(\varepsilon+1)a} \le e^{2a};$$

that is, π_ε is a Harnack density with constants $H = e^{2a}$ and τ which are independent of ε.

Finally, since Ω_ε is non-complete, we can ask whether or not π_ε is a Koebe density. Since $\pi_\varepsilon(x) = \rho_\varepsilon(x)/d(x)$, we see from (5.3) that π_ε will be a Koebe density, with constant $K = (2e^{\varepsilon\kappa} - 1)/\varepsilon$ (assuming $\varepsilon e \le 1$), provided (Ω, k) is roughly κ-starlike (and (Ω, d) uniformly locally quasiconvex).

These conditions describing when π_ε will be a Harnack or Koebe density do not require that (Ω, k) be Gromov hyperbolic. We record the above information for later reference; see also Lemma 2.8 and Corollary 5.8. Note too that (Ω, d_ε) is bounded with $\operatorname{diam}_\varepsilon \Omega_\varepsilon \le 2/\varepsilon$.

5.4. Lemma. *Let (Ω, d) be a locally a-quasiconvex abstract domain and fix a base point $w \in \Omega$. Then for any $\varepsilon > 0$,*

$$\pi_\varepsilon(x) = \rho_\varepsilon(x)/d(x) = d(x)^{-1}\exp(-\varepsilon k(x,w))$$

is a Harnack density with constant $H = e^{2a}$. If in addition (Ω, k) is roughly κ-starlike, then π_ε is also a Koebe density, with constant $K = \max\{\varepsilon e, (2e^{\varepsilon\kappa} - 1)/\varepsilon\}$.

The above, in conjunction with Lemma 2.11 and Proposition 5.6(b), provides a one-to-one correspondence between conformal metric-densities on Ω and the same on Ω_ε. Here is a precise statement of this.

5.5. Corollary. *Suppose (Ω, d, μ) is an abstract domain having bounded geometry and a Gromov hyperbolic roughly starlike quasihyperbolization (Ω, k). Let $\Omega_\varepsilon = (\Omega, d_\varepsilon, \mu_\varepsilon)$ be the deformation of Ω via the density π_ε defined just above. If σ is a conformal density on Ω_ε, then its pull-back $\rho = \sigma\,\pi_\varepsilon$ is a conformal density on Ω, and conversely if ρ is a conformal density on Ω, then its push-forward $\sigma = \rho\,\pi_\varepsilon^{-1}$ is a conformal density on Ω_ε. In both cases $\Omega_\rho = (\Omega_\varepsilon)_\sigma$, and the metric-density parameters depend only on each other and the data associated with Ω.*

5.C. Bounded Geometry and its Consequences. Recall our definition that an abstract domain (Ω, d, μ) is a locally complete, rectifiably connected, non-complete metric measure space; these are our standing metric hypotheses. We say that (Ω, d, μ) has *bounded Q-geometry*, $Q > 1$, provided it is both *locally upper Ahlfors Q-regular* (see §2.J) and *weakly locally Q-Loewner*; this latter condition means that there exists a positive constant m such that for all $x \in \Omega$, and all non-degenerate disjoint continua E, F in $\lambda B(x)$,

$$\Delta(E,F) \le 16 \implies \quad \operatorname{mod}_Q(E,F;\Omega) \ge m.$$

Any space Ω with the property that Whitney balls $\lambda B(x)$ (with a fixed parameter) are uniformly bilipschitz equivalent to Euclidean balls (in a fixed dimension) is easily seen to have bounded geometry. Other examples include Riemannian manifolds with Ricci bounded geometry as well as the exotic examples of Bourdon-Pajot and Laakso for any $Q > 1$; see [BHK01, Exs.9.7, p.86] and the references mentioned there. (Note that by Proposition 5.6, (Ω, d, μ) has bounded geometry if and only if its quasihyperbolization satisfies the condition studied in [BHK01, Chpt.9].)

The first part of bounded Q-geometry is a necessary condition for Ω to support any conformal density (at least when Ω is locally quasiconvex). The second part of bounded Q-geometry, the weak local Loewner criterion, can also be described in terms of Poincaré inequalities as explained in [BHK01, Proposition 9.4] and [HK98, §5]; it ensures that there are plenty of curves available (e.g., it gives local quasiconvexity). However, to substantiate this existence of many curves requires the use of certain modulus estimates (see Facts 2.15,2.16), and these estimates in turn require an upper mass condition.

The Loewner part of bounded Q-geometry also performs an essential role in two other places. First, it is a key player in the proof of Proposition 2.9, which is the crucial ingredient in the proof of (d) implies (e) in Theorem D. Second, for uniform Loewner spaces with appropriately 'thick' boundaries, the Koebe condition for a metric-density follows from the Harnack and Ahlfors condition; this fact is utilized in the proof of Theorem C.

Bounded geometry provides a number of essential properties for our underlying space.

5.6. Proposition. *Let (Ω, d, μ) be an abstract domain having bounded Q-geometry (with constants M, m, λ). Then:*

(a) *μ is locally Ahlfors Q-regular.*
(b) *Ω is locally quasiconvex with constants which depend only on Q, M, m, λ.*
(c) *Ω is locally Q-Loewner with a control function ψ and parameters κ, ε_0 which depend only on the data Q, M, m, λ associated with Ω.*

Here is a useful consequence of the above.

5.7. Corollary. *Let ρ be a Harnack Koebe density on an abstract domain (Ω, d, μ) having bounded Q-geometry. Then Ω_ρ is locally Ahlfors Q-regular and locally Q-Loewner with parameters and a control function which depend only on the data associated with ρ and Ω.*

Note that if Ω_ρ above is also uniform, then it would be globally Loewner by [BHK01, Theorem 6.4]. We record the following consequence of this observation.

5.8. Corollary. *Let (Ω, d, μ) be an abstract domain having bounded Q-geometry (with constants M, m) and a Gromov δ-hyperbolic roughly κ-starlike quasihyperbolization (Ω, k). Then any BHK-uniformization $\Omega_\varepsilon = (\Omega, d_\varepsilon, \mu_\varepsilon)$ of Ω is a* *bounded locally Ahlfors Q-regular locally Q-Loewner space (hence of bounded Q-geometry), and even Q-Loewner when $\varepsilon < \varepsilon(\delta)$. Here the various parameters and control functions depend only on $\delta, \kappa, Q, M, m, \lambda, \varepsilon$.*

5.D. Lifts and Metric Doubling Measures. Here we discuss the ideas behind Theorem C and briefly outline the proof. Recall the notion of a metric doubling measure discussed near the end of §2.F.

We define the *lift* ρ_ν of ν via the formula

$$\rho_\nu(x) := \frac{\nu(\Sigma_x)^{1/P}}{d(x)}\,; \tag{5.9}$$

here ν is a P-dimensional metric doubling measure on $\partial_G\Omega$ and, for some fixed base point $w \in \Omega$, the *shadow* of a point $x \in \Omega$ is

$$\Sigma_x := \{\zeta \in \partial_G\Omega : k(x, [w,\zeta]) \le R\}.$$

It is not hard to see that there are constants $R = R(\delta, \kappa, \varepsilon)$ and $C = C(\delta, \kappa, \varepsilon)$ such that

$$S_x := \partial_\varepsilon\Omega \cap 2B_\varepsilon(x) \subset \Sigma_x \subset \partial_\varepsilon\Omega \cap CB_\varepsilon(x); \tag{5.10}$$

of course we are using the natural identification of $\partial_G\Omega$ with $\partial_\varepsilon\Omega$. Employing (5.10), the doubling property of ν, and the fact that π_ε is Koebe, it is straightforward to verify that the push-forward of ρ_ν (as defined in (5.9)) via the uniformizing density π_ε gives a density on Ω_ε which is bilipschitz equivalent to the density defined via the formula

$$\rho(x) := \frac{\nu(S_x)^{1/P}}{d_\varepsilon(x)} \qquad \text{where } S_x := \partial_\varepsilon\Omega \cap 2B_\varepsilon(x).$$

Theorem C now follows from Corollary 5.5 once we verify that ρ is a Borel Harnack Ahlfors Koebe doubling metric-density on Ω_ε. $(\ddot\smile)$ The first two of these are easy. The Koebe property follows from Theorem E once we know the Ahlfors volume growth property. To see that Theorem E can be applied, we first use Fact 2.2 along with Theorem F to see that the Whitney ball modulus property holds.

To establish the Ahlfors property we need the doubling property. This in turn requires the following 'quasihyperbolic doubling' result.

5.11. Proposition. *Let (Ω, d, μ) be a locally a-quasiconvex locally Ahlfors Q-regular abstract domain. Then its quasihyperbolization (Ω, k) is locally doubling in the sense that if $\Sigma \subset \Omega$ is a set of points satisfying*

$$0 < t \le k(x,y) \le T < \infty \qquad \text{for all } x, y \in \Sigma,\ x \ne y,$$

then the cardinality of Σ is bounded by

$$\#\Sigma \le 2M^2 8^Q (2M^2 24^Q)^{8aT/\lambda t}.$$

Here M is the local regularity constant and λ the Whitney ball constant.

Here is an interesting application of Theorems C and D which provides a characterization of doubling conformal densities in terms of metric doubling measures. We say that a metric-density ρ on Ω is *induced by a metric doubling measure* ν on $\partial_G\Omega$ if there exists a constant $C \geq 1$ such that

$$C^{-1}\rho_\nu(x) \leq \rho(x) \leq C\rho_\nu(x) \qquad \text{for } \mu\text{-a.e. } x \in \Omega.$$

5.12. Theorem. *Assume the basic minimal hypotheses, that the Gromov boundary of Ω is uniformly perfect, and $P < Q$. A metric-density ρ on Ω is a doubling conformal density, with $(\partial_\rho\Omega, d_\rho)$ Ahlfors P-regular, if and only if ρ is induced by some P-dimensional metric doubling measure on $\partial_G\Omega$.*

5.E. Volume Growth Problem. Here we briefly outline the proof of Theorem A. Recall that this result provides our answer to the problem of deciding when there exists a uniformizing conformal density (so, in particular, the associated measure (2.7) should have Ahlfors regular volume growth). The necessity in this result follows from Fact 4.1 along with Fact 2.20. The real work involved is in establishing the sufficiency.

The first major step is to prove Theorem D. Following [BHR01], we say that a metric-density ρ, on a *uniform* space (Ω, d, μ), is *doubling* provided μ_ρ is a doubling measure on (Ω, d); i.e., there exists a constant $D = D_\rho$ such that for all $x \in \Omega$,

$$\text{(D)} \qquad \mu_\rho[B(x;2r)] \leq D\,\mu_\rho[B(x;r)] \qquad \text{for all } r > 0.$$

When Ω is not uniform, the above doubling condition may fail to hold even if ρ is 'nice'; e.g., consider $\rho = |f'|$ on $\Omega = \{x + iy : |y| < 1\}$ in the complex plane, where f is a conformal homeomorphism of Ω onto the unit disk. To compensate for this we employ the following definition: a conformal density ρ, on an abstract domain (Ω, d, μ) with Gromov hyperbolic quasihyperbolization, is *doubling* if its push-forward is doubling on some BHK-uniformization Ω_ε of Ω. (Since all such spaces Ω_ε are QS equivalent, and doubling measures can be defined for conformal gauges, there is no ambiguity here. See §2.C for a discussion of conformal gauges, and also the very last paragraph of §2.F.)

Notice that the doubling condition (D) uses d-balls but μ_ρ-measure and thus interweaves the measure properties of the deformed space with the metric properties of the original (or uniformized) space.

Our proof of Theorem D requires the Gehring-Hayman Inequality (Theorem B), Corollary 2.10, and the following two results. First we give a necessary condition for a metric-density to be doubling.

5.13. Proposition. *Let ρ be a Harnack density on an a-uniform locally Ahlfors Q-regular space (Ω, d, μ) (with constants H, M, λ). Suppose that ρ is also doubling on Ω (with constant D). Then quasihyperbolic geodesics in Ω are double c-cone arcs in Ω_ρ where $c = c(D, H, M, Q, a, \lambda)$. If in addition Ω is bounded, then so is Ω_ρ and $\partial\Omega \subset \partial_\rho\Omega$, with equality holding when Ω is also locally Q-Loewner.*

Next we state a sufficient condition for a metric-density to be doubling.

5.14. Proposition. *Let ρ be a conformal density on an a-uniform locally lower Ahlfors Q-regular space (Ω, d, μ) (with constants H, A, K, M, λ). Suppose that quasihyperbolic geodesics in Ω are double c-cone arcs in Ω_ρ. Then Ω_ρ is Ahlfors Q-regular with a parameter which depends only on c and the data $H, A, K, M, Q, a, \lambda$ associated with ρ and Ω. If in addition Ω and Ω_ρ are both bounded, then ρ is doubling on Ω with a parameter which depends only on the aforementioned data and the quantity q given in the proof of Lemma 3.3.*

The second major step in the proof of Theorem A is to establish Theorem C. Its proof is outlined above in §5.D. That done, we use Theorem C to obtain a doubling conformal density, which by Theorem D is also bounded and uniformizing.

6. Characterizations of Uniform Spaces

Here we mention a number of characterizations for uniform spaces and uniform domains. In [Väi88] Väisälä provides a complete description of the various possible twisted double cone conditions (which he calls *length cigars, diameter cigars, distance cigars,* and *Möbius cigars*). The work [Mar80] of Martio should also be mentioned.

6.A. Metric Characterizations. We have already mentioned that uniform spaces are precisely the abstract domains in which the quasihyperbolic metric is bilipschitz equivalent to the so-called j metric; see Fact 3.2.

It turns out that the following seemingly weaker quasihyperbolic metric condition also characterizes uniform spaces. For uniform subdomains of Banach spaces this result is due to Väisälä [Väi91, 6.16, 6.17]. Here we write

$$r(x, y) := \frac{|x - y|}{d(x) \wedge d(y)}$$

to denote the so-called *relative distance* between x, y. See [BH07] for the following version.

6.1. Theorem. *A locally quasiconvex abstract domain is uniform if and only if there is a homeomorphism $\vartheta : [0, \infty) \to [0, \infty)$ satisfying $\limsup_{t\to\infty} \vartheta(t)/t < 1$, and such that for all points x, y, $k(x, y) \le \vartheta(r(x, y))$. The uniformity constant depends only on ϑ, and conversely in an a-uniform space, one can always take $\vartheta(t) = b \log(1 + t)$ with $b = b(a)$.*

6.B. Gromov Boundary Characterizations. We call $\Omega \subsetneq \mathbb{R}^n$ a *Gromov domain* if its quasihyperbolization (Ω, k) is Gromov hyperbolic. Bonk, Heinonen and Koskela corroborated the following [BHK01, Proposition 7.12].

6.2. Fact. *A Gromov domain in Euclidean space is uniform if and only if it is linearly locally connected.*

They utilized the above to establish the following [BHK01, Theorem 7.11].

6.3. Fact. *A (bounded) Gromov domain in Euclidean space is uniform if and only if the canonical gauge on the Gromov boundary is quasisymmetrically equivalent to the Euclidean gauge on the Euclidean boundary.*

The careful reader will recognize that the Bonk-Heinonen-Koskela results were established for regions on the sphere (i.e., using the spherical metric). Results of Balogh and Buckley [BB06] are useful in this regard.

Väisälä has recently proven a Banach space analog of the above result; see [Väi05b]. Along with providing a dimension free version of this result, he also considers arbitrary domains (not just bounded) and replaces QS equivalence with QM equivalence. In addition, he provides an example of a Gromov hyperbolic domain which is LLC but not uniform.

6.C. Characterizations using QC Maps. It is evident that bilipschitz homeomorphisms map uniform spaces to uniform spaces. This also holds true for QS and QM maps of uniform domains in Euclidean space and in Banach spaces [Väi99, Theorem 10.22], but not in the general metric space setting, and not for QC maps. On the other hand, according to [BKR98, 2.4], the average derivative of a quasiconformal map $f : \mathbb{B}^n \to \Omega \subset \mathbb{R}^n$ is a conformal metric density on $\mathbb{B}^n$ (a uniform n-Loewner n-regular space). Thus we can appeal to Theorem D and read off a number of conditions which characterize when Ω will be uniform.

6.D. Capacity Conditions. It is known that given $0 < \lambda \leq 1/2$, there exists a constant $c = c(\lambda, n) > 0$ such that

$$\forall\, x, y \in \Omega : \quad k(x, y) \geq 2 \quad \Longrightarrow \quad \operatorname{mod}(\lambda\bar{B}(x), \lambda\bar{B}(y); D) \geq c/k(x, y)^{n-1};$$

this is valid for any proper subdomain Ω of $\mathbb{R}^n$. To prove it, one starts by using Lemma 2.14 to select an appropriate cover of any quasihyperbolic geodesic joining x, y, and then a standard application of the Poincaré inequality applied to adjacent balls leads to the asserted inequality. See the proof of [HK96, Theorem 6.1].

Let $C > 0$ and $0 < \lambda \leq 1/2$. A proper subdomain Ω of $\mathbb{R}^n$ is a (C, λ)-*k-cap domain* provided

$$\forall\, x, y \in D : \quad k(x, y) \geq 2 \quad \Longrightarrow \quad \operatorname{mod}(\lambda\bar{B}(x), \lambda\bar{B}(y); D) \leq C/k(x, y)^{n-1}.$$

Thus in a k-cap domain D, we have $\operatorname{mod}(\lambda\bar{B}(x), \lambda\bar{B}(y); D) \simeq k(x, y)^{1-n}$ for points with $k(x, y) \geq 2$, with constants of comparison dependent only on λ, n, and the k-cap parameter.

This is the two-sided version of a condition introduced by Buckley in [Buc04] to study quasiconformal images of domains which satisfy a quasihyperbolic boundary condition. As explained on p.26 of that paper, a (C, λ)-k-cap condition implies a (C', λ')-k-cap condition for some $C' = C'(C, \lambda, \lambda', n)$. We mainly consider the case $\lambda = 1/2$, and refer to a $(C, 1/2)$-k-cap domain simply as a C-*k-cap domain.*

Every uniform domain in $\mathbb{R}^n$ is a k-cap domain, and the class of k-cap domains is invariant under quasiconformal mappings (with a quantitative change of parameter C). For proofs of these statements see [Buc04].

Recently we established the following characterization for uniform domains in Euclidean space; see [BH06, Theorem 3.5].

6.4. Theorem. *A proper subdomain of $\mathbb{R}^n$ is uniform if and only if it is both QED_{wb} and a k-cap domain.*

This result is quantitative.

6.E. LLC and Slice Conditions. By utilizing certain slice conditions, Balogh and Buckley [BB03] established a number of geometric characterizations for Gromov hyperbolic spaces. Here we mention the following new characterization of uniform spaces; see [BH07]

6.5. Theorem. *An abstract domain is uniform and LEC if and only if it is quasiconvex, LLC_2 with respect to arcs, and satisfies a weak slice condition.*

These implications are quantitative.

We have modified the usual LLC conditions (see §2.E) by requiring that the points in question be joinable by rectifiable arcs (rather than just by continua). Every uniform domain in $\mathbb{R}^n$ is LLC. In fact every uniform space is quasiconvex and thus LLC_1 with respect to arcs. However, uniform spaces need not be LLC_2; e.g., an 'asterik' type space (the disjoint union of a point and a bunch of line segments or rays, with its intrinsic length distance) may be uniform but not LLC_2. We say that a locally complete metric space is *locally externally connected*, abbreviated LEC, provided there is a constant $c \geq 1$ such that the (LLC_2) condition holds for all points $x \in \Omega$ and all $r \in (0, d(x)/c)$.

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