# FRACTAL INNER CHORDARC DISKS

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Abstract. We introduce a class of fractal inner chordarc domains and characterize them as inner Ahlfors regular John disks. We analyze 'global' parametrizations for such regions, describe the associated Riemann maps and exhibit several illustrative examples.

## 1. Introduction

A Jordan curve $C$ enjoys a *chordarc* condition provided there is a constant $c \geq 1$ so that for all points $x, y$ in $C$, some subarc $A$ of $C$ joining $x, y$ has arclength

$$\ell(A) \leq c|x-y|.$$

Since $\ell(A) \geq \operatorname{diam}(A)$ for any arc $A$, chordarc curves are bounded turning (so quasicircles in the plane—see Section 2 for basic definitions, terminology and notation); they are also locally rectifiable, and, in fact, are *Ahlfors regular* meaning there is a constant $a \geq 1$ such that for all points $z$ and all $r > 0$,

$$\ell(C \cap B(z;r)) \leq a\, r.$$

That Ahlfors regular quasicircles are chordarc is true and part of folklore; see [Pom92, Prop7.7,p.163]. In [GH98] we extended these notions to fractal type curves which have arbitrary Hausdorff dimension $\alpha > 1$ and verified, e.g., that the $\alpha$-dimensional chordarc curves are precisely the bounded turning $\alpha$-dimensional Ahlfors regular curves. The current article should be considered a 'one-sided' version of [GH98].

Väisälä introduced the notion of *inner chordarc* plane domains [Väi87] and established that a cylindrical domain $G \times \mathbf{R}$ in $\mathbf{R}^3$ is quasiconformally equivalent to a ball if and only if the base domain $G \subset \mathbf{R}^2$ is

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inner chordarc [Väi89]. Ghamsari studied these domains and corroborated that inner chordarc domains are exactly the John disks with Ahlfors regular boundary [Gha90].

Here we combine these generalizations. We introduce the classes of *$\alpha$-dimensional inner chordarc* and *$\alpha$-dimensional inner Ahlfors regular* domains. Roughly speaking, $G$ is $\alpha$-dimensional inner chordarc provided a chordarc-like condition holds for the boundary $C = \partial G$ where now we require that the $\mathcal{H}^\alpha$-measure of $A$ be comparable to the inner distance between its endpoints $x, y$, and $G$ is $\alpha$-dimensional inner Ahlfors regular if the $\mathcal{H}^\alpha$-measure of that part of $C$ inside an inner ball of radius $r$ is comparable to $r^\alpha$. One of our results is the following analog of the above mentioned folklore fact.

**Theorem A.** *For a Jordan plane domain $G$, the following are quantitatively equivalent.*

(a) *$G$ is an $\alpha$-dimensional inner chordarc disk.*
(b) *$G$ is an $\alpha$-dimensional inner Ahlfors regular John disk.*
(c) *$G$ is a John disk and $\mathcal{H}^\alpha(A)$ is comparable to $(\operatorname{diam} A)^\alpha$ for each subarc $A \subset \partial G$.*

Here the phrase 'quantitatively equivalent' means that the stated conditions all imply each other (of course) and that the relevant constants depend only on each other. We shall see that it suffices to know the 'measure/diameter' condition in (c) above only for certain boundary subarcs; see Lemma 2.2

A moments thought reveals that chordarc curves are precisely those curves which have bilipschitz parametrizations; the analog of this holds for $\alpha$-dimensional chordarc curves. We can also describe the fractal inner chordarc regions in terms of certain parametrizations of their boundaries. We let $\mathbf{U}$ denote either the unit disk when $G$ is bounded or the upper half-plane when $G$ is unbounded.

**Theorem B.** *A Jordan plane domain $G$ is $\alpha$-dimensional inner chordarc if and only if $(\partial G, \rho)$ is bilipschitz equivalent to $(\partial\mathbf{U}, d^{1/\alpha})$; $d$ is euclidean distance and $\rho$ is any metric on $\partial G$ satisfying $\delta_G \le \rho \le \lambda_G$.*

Again, the relevant constants depend only on each other. An immediate corollary is that two fractal inner chordarc domains, which are either both bounded or both unbounded, are bilipschitz equivalent with respect to their respective inner distances if and only if they have the same dimension.

Another important property of chordarc curves, established by Tukia [Tuk81], and independently by Jerison and Kenig [JK82, 1.13, p.227], and also by Latfullin [Lat82], is that they are bilipschitz equivalent to a circle or a line via a global homeomorphism of the plane. Väisälä verified the analogous property for inner chordarc domains showing that they are the images of disks or half-planes under locally bilipschitz homeomorphisms [Väi87]. We establish the following fractal version of these results.

**Theorem C.** *A Jordan plane domain $G$ is $\alpha$-dimensional inner chordarc if and only if there is a homeomorphism $\varphi : \mathbf{U} \to G$ satisfying*

$$\partial\varphi(\zeta) \approx \operatorname{dist}(\zeta, \partial\mathbf{U})^{1/\alpha-1} \qquad \text{for all } \zeta \in \mathbf{U}.$$

Here we write

$$\overline{\partial}\varphi(\zeta) = \limsup_{w\to\zeta} \frac{|\varphi(w)-\varphi(\zeta)|}{|w-\zeta|} \quad , \quad \underline{\partial}\varphi(\zeta) = \liminf_{w\to\zeta} \frac{|\varphi(w)-\varphi(\zeta)|}{|w-\zeta|}$$

and the stated condition means that there is a constant $M$ so that

$$M^{-1}\operatorname{dist}(\zeta,\partial\mathbf{U})^{1/\alpha-1} \le \underline{\partial}\varphi(\zeta) \quad \text{and} \quad \overline{\partial}\varphi(\zeta) \le M\ \operatorname{dist}(\zeta,\partial\mathbf{U})^{1/\alpha-1}.$$

Again the result is quantitative.

Pommerenke [Pom82, Thm1(ii)], [Pom92, Thm5.2(ii)] characterized bounded John disks in terms of their Riemann maps. His condition can be described as follows: We associate to each non-zero point $\zeta$ of the unit disk the smaller subarc $I$ of the unit circle which is cut off by the circular arc through $\zeta$ perpendicular to the unit circle with $\zeta$ its arclength midpoint. Then the image of the disk is a John disk if and only if the image of every such boundary arc $I$ has diameter comparable to the distance from the image of the point $\zeta$ to the domain's boundary. Ghamsari [Gha90, Thm2.19] generalized Pommerenke's result to inner chordarc domains by replacing diameter with length. See Fact 2.10.

Pommerenke's result remains true for unbounded domains too, provided the unit disk is replaced by a half-plane. We consider an $\alpha$-dimensional analog of the Pommerenke-Ghamsari condition (obtained by replacing diameter or length with Hausdorff measure) and establish the following.

**Theorem D.** *A Jordan plane domain $G$ is $\alpha$-dimensional inner chordarc if and only if $\mathcal{H}^\alpha(fI_\zeta) \approx \operatorname{dist}(f\zeta,\partial G)^\alpha$ for each $\zeta \in \mathbf{U}$, where $f : \mathbf{U} \to G$ is conformal with a homeomorphic extension $f : \bar{\mathbf{U}} \to \bar{G}$.*

Again the result is quantitative.

This document is organized as follows: Section 2 contains preliminary information including basic definitions and terminology descriptions as well as elementary and/or well-known facts. In Sections 3, 4 we examine $\alpha$-dimensional inner chordarc curves and their parametrizations, and we prove Theorems A, B, C. Section 5 is devoted to explaining and corroborating Theorem D. We conclude with Section 6 where we exhibit numerous examples which illustrate our results.

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## 2. Preliminaries

2.A. **General Information.** Our notation is relatively standard and conforms with that of [GH98]. We work in the euclidean plane $\mathbf{R}^2$ and let $d(x, y) = |x - y|$ be the usual euclidean distance. We let $B(x; r) = \{y : |x - y| < r\}$ and $S(x; r) = \partial B(x; r) = \{y : |x - y| = r\}$ denote the open disk and circle of radius $r$ centered at the point $x$. We put $\mathbf{D} = B(0; 1)$ (the unit disk) and let $\mathbf{H}$ be the upper half-plane. We write $c = c(a, \dots)$ to indicate a constant $c$ which depends only on the parameters $a, \dots,$ and $a \approx b$ means there exists a positive finite constant $c$ with $1/c \le a/b \le c$. Typically $a, b, c$ will be constants which depend on various parameters, and we try to make this as clear as possible often giving explicit values. We reserve the letter $k$ for an absolute constant, whose actual value may vary even in the same line.

Throughout this paper $G$ denotes a Jordan domain in the plane $\mathbf{R}^2$; thus $G$ is simply connected and its boundary $C = \partial G$ is a Jordan curve (either compact, if $G$ is bounded, or 'going through' infinity otherwise). Thus there is a conformal map from $\mathbf{U}$ onto $G$, which has a homeomorphic extension mapping $\bar{\mathbf{U}}$ onto $\bar{G}$ and induces a homeomorphism from $\mathbf{T} = \partial \mathbf{U}$ onto $C$, where $\mathbf{U}$ is either the unit disk $\mathbf{D}$ when $G$ is bounded (then $\mathbf{T} = \mathbf{S}^1$) or the upper half-plane $\mathbf{H}$ when $G$ is unbounded (then $\mathbf{T} = \mathbf{R}$).

We call $\gamma$ an *arc* if it is the homeomorphic image of an interval; we use the adjectives closed or open to indicate whether $\gamma$ does or does not contain its endpoints. When $x, y$ are points on an arc $\gamma$ we write $\gamma(x, y)$ to denote the subarc of $\gamma$ between $x$ and $y$. Given two points $x, y$ on $C = \partial G$, we let $C(x, y)$ and $C^\alpha(x, y)$ denote the subarcs of $C$ joining $x, y$ with minimal diameter and with minimal $\mathcal{H}^\alpha$-measure respectively. Thus $C(x, y)$, $C^\alpha(x, y)$ are respectively the smaller, 'shorter' subarcs of $C$ joining $x, y$. (When $C$ is unbounded, there is a unique subarc of $C$ joining $x, y$. If $C$ is bounded and both subarcs joining $x, y$ have the same 'size', we just select one.)

The following topological fact will be useful.

**2.1. Lemma.** *Let $C = \partial G$ with $G$ a Jordan plane domain. Put $C_0 = B_0 \cap C$ where $B_0 = B(w_0; r)$ and $w_0 \in C$. Assume there is some point $w \in C \setminus \bar{B}_0$. Then there exist disjoint arcs $\delta_k \subset \partial B_0 \cap G$ with endpoints $x_k, y_k \in C$ such that $C_0 \subset \bigcup \alpha_k$, where $\alpha_k$, $\beta_k$ are the components of $C \setminus \{x_k, y_k\}$ with $w \in \beta_k$.*

*Proof.* First note that there are countably many components of $C_0$, since these are all open subarcs of $C$. Let $\gamma_1$ be any such component and fix a point $w_1 \in \gamma_1$. Then $w_1, w$ are separated (in $G$) by the circle $\partial B_0$, so a modification of [New51, VI.7.1,p.151] furnishes a component $\delta_1$ of $\partial B_0 \cap G$ which separates $w_1, w$ in $G$. Thus $\delta_1$ is a subarc of the circle $\partial B_0$ with endpoints $x_1, y_1 \in C$. We also know that $x_1, y_1$ separate $w_1, w$ on $C$, so we can label the components $\alpha_1, \beta_1$ of $C \setminus \{x_1, y_1\}$ so that $\gamma_1 \subset \alpha_1$ and $w \in \beta_1$. See [Gha90, 2.9,2.11]. (Note that in general $\gamma_1 \neq \alpha_1$ and in fact $\alpha_1$ may contain many components of $C_0$.)

If $C_0 \subset \alpha_1$, we are finished. Suppose $C_0 \not\subset \alpha_1$. Then there exists a component $\gamma_2$ of $C_0$ with $\gamma_2 \cap \alpha_1 = \emptyset$. We repeat the above argument to obtain a component $\delta_2$ of $\partial B_0 \cap G$ which separates $w_2 \in \gamma_2$ and $w$ in $G$ with endpoints $x_2, y_2 \in C$ and where the components $\alpha_2, \beta_2$ of $C \setminus \{x_2, y_2\}$ are labeled so that $\gamma_2 \subset \alpha_2$ and $w \in \beta_2$. If $C_0 \subset \alpha_1 \cup \alpha_2$, we are done; otherwise we continue the above process. $\square$

Note that in the above notation we have $\operatorname{diam} \delta_k \leq 2r$ for all $k$; in fact, since the $\delta_k$ are disjoint subarcs of the circle $\partial B_0$, we find that

$$\sum_k \operatorname{diam} \delta_k \leq \sum_k \ell(\delta_k) \leq 2\pi r.$$

Recall that *Hausdorff $\alpha$-dimensional measure*, $\mathcal{H}^\alpha$, is given by

$$\mathcal{H}^\alpha(A) = \lim_{r \to 0} \left[ \inf \left\{ \sum \operatorname{diam}(U_i)^\alpha : A \subset \cup U_i, \operatorname{diam}(U_i) \leq r \right\} \right].$$

We often utilize an $\mathcal{H}^\alpha / \operatorname{diam}$ condition such as

$$a^{-1} (\operatorname{diam} A)^\alpha \leq \mathcal{H}^\alpha(A) \leq a\, (\operatorname{diam} A)^\alpha \tag{HD}$$

where $A$ is a subarc of a Jordan curve $C$. In light of this, the following information proves useful.

**2.2. Lemma.** *For any compact Jordan curve $C$, the following are equivalent:*

(a) (HD) *holds for every subarc $A$ of $C$.*
(b) (HD) *holds for $A = C^\alpha(x, y)$ for all $x, y \in C$.*
(c) (HD) *holds for $A = C(x, y)$ for all $x, y \in C$.*
(d) (HD) *holds for $A = \varphi(I)$ for every subarc $I \subset \mathbf{S}^1$ with $\ell(I) \leq \pi$, where $\varphi : \mathbf{S}^1 \to C$ is any homeomorphism.*

*Here the constant $a$ will vary from* (a) *to* (d), *but $\alpha$ is fixed.*

*Proof.* First we point out that if any of these conditions hold, $C$ has $\sigma$-finite $\mathcal{H}^\alpha$-measure, and when $C$ is bounded (HD) holds for $A = C$ provided we change the right-hand constant to $2a$. Next, as (a) clearly implies each of (b), (c), (d), it suffices to establish the converses. Let $A$ be an arbitrary subarc of $C$ with endpoints $x, y$. If $A$ is an arc of the given form, there is nothing to prove, so we may assume that $A$ is the 'other' subarc.

(c)$\Rightarrow$(a) We assume $A$ is the 'bigger' subarc of $C$ joining $x, y$. Choose $z \in A$ with $\operatorname{diam} A(x,z) = \operatorname{diam} A(y,z)$. Then $C(x,z) = A(x,z)$ and $C(y,z) = A(y,z)$. Therefore

$$\mathcal{H}^\alpha(A) = \mathcal{H}^\alpha(A(x,z)) + \mathcal{H}^\alpha(A(y,z)) \le 2a(\operatorname{diam} A)^\alpha$$

and

$$\mathcal{H}^\alpha(A) = \mathcal{H}^\alpha(A(x,z)) + \mathcal{H}^\alpha(A(y,z)) \ge 2a^{-1}(\frac{1}{2}\operatorname{diam} A)^\alpha,$$

so (HD) holds with the new constant $a\max\{2, 2^{\alpha-1}\}$.

(b)$\Rightarrow$(a) We assume $A$ is the 'longer' subarc of $C$ joining $x, y$. Choose $z \in A$ with $\mathcal{H}^\alpha(A(x,z)) = \mathcal{H}^\alpha(A(y,z))$. Then $C^\alpha(x,z) = A(x,z)$ and $C^\alpha(y,z) = A(y,z)$. Therefore

$$\mathcal{H}^\alpha(A) = \mathcal{H}^\alpha(A(x,z)) + \mathcal{H}^\alpha(A(y,z)) \le 2a(\operatorname{diam} A)^\alpha.$$

On the other hand,

$$\mathcal{H}^\alpha(A) \ge \frac{1}{2}\mathcal{H}^\alpha(C) \ge \frac{1}{2a}(\operatorname{diam} C)^\alpha \ge \frac{1}{2a}(\operatorname{diam} A)^\alpha,$$

so (HD) holds with the new constant $2a$.

(d)$\Rightarrow$(a) We assume $A = \varphi(J)$ where $J$ is the complement of a small subarc $I$ of the unit circle. Let $A_k = \varphi(J_k)$ where $J_1, J_2$ are equal size subarcs with $J = J_1 \cup J_2$. Then

$$\mathcal{H}^\alpha(A) = \mathcal{H}^\alpha(A_1) + \mathcal{H}^\alpha(A_2) \le 2a(\operatorname{diam} A)^\alpha$$

and

$$\mathcal{H}^\alpha(A) \ge \max_{k=1,2} \mathcal{H}^\alpha(A_k)) \ge \max_{k=1,2} a^{-1}(\operatorname{diam} A_k)^\alpha \ge \frac{1}{2^\alpha a}(\operatorname{diam} A)^\alpha,$$

so (HD) holds with the new constant $2^\alpha a$. $\square$

2.B. **Inner Distances.** In his investigation of inner chordarc domains, Väisälä utilized the *inner (length) distance* which is defined, relative to $G$, by

$$\lambda(x,y) = \lambda_G(x,y) = \inf\{\ell(\gamma) : \gamma \text{ joins } x, y \text{ in } G\}.$$

It is easy to see that $\lambda_G$ is a distance function on $G$; however, in general $\lambda_G$ need not be a metric on $\bar{G}$, (it is a distance where finite) and even when it is it need not be continuous with respect to the euclidean metric. Because of the pathology associated with $\lambda_G$, it is more convenient to work with the *inner (diameter) distance* relative to $G$ defined by

$$\delta(x, y) = \delta_G(x, y) = \inf\{\operatorname{diam}(\gamma) : \gamma \text{ joins } x, y \text{ in } G\}.$$

It is not difficult to see that $\delta_G$ is a distance function on $\bar{G}$ which is continuous with respect to the euclidean metric and that

$$\operatorname{diam}_\delta(A) = \operatorname{diam}(A) \qquad \text{for each subarc } A \subset \partial G.$$

Moreover, we have the following useful result from [GNV94, 5.14]

**2.3. Fact.** *The inner distances $\delta_G$ and $\lambda_G$ are comparable in a Jordan John disk, meaning that $\delta_G(x, y) \le \lambda_G(x, y) \le b\,\delta_G(x, y)$ for all $x, y \in \bar{G}$ where $b$ depends only on the John constant.*

2.C. **John Disks.** A domain $D$ is a *John domain* provided there is a constant $b \ge 1$ with the property that each pair of points $x, y$ in $D$ can be joined by a rectifiable arc $\gamma$ in $D$ which satisfies

$$\min\{\ell(\gamma(x, z)), \ell(\gamma(y, z))\} \le b\operatorname{dist}(z, \partial D) \quad \text{for all } z \in \gamma. \qquad \text{(J)}$$

This condition ensures that $D$ contains the twisted (or curvilinear) double $b$-cone $\cup\{B(z; \rho(z)/b) : z \in \gamma\}$ where $\rho(z)$ denotes the left-hand-side of (J). We call $\gamma$ a *double $b$-cone arc* joining $x$ and $y$. These domains were first studied by John [Joh61] and Martio and Sarvas [MS79]. For equivalent definitions, many properties, and other references see also the work of Näkki and Väisälä [NV91], as well as that of Gehring, Hag, Martio [GHM89], Pommerenke [Pom82], [Pom92, Chpt5] and Ghamsari, Näkki, Väisälä [GNV94]. Actually the above definition is based on so-called 'length' cones. One can just as well use 'distance' and/or 'diameter' cones, facts which are well-documented in [NV91].

We are concerned with the so-called *John disks* which are the simply connected plane John domains; in fact we only consider Jordan John disks. There are several alternative descriptions for John disks as chronicled by Näkki and Väisälä [NV91, 4.5,6.3]. We require only the following characterizations for our purposes.

**2.4. Fact.** *For a Jordan domain $G$ in $\mathbf{R}^2$, the following are quantitatively equivalent.*

(a) *$G$ is a $b$-John disk.*
(b) $\operatorname{diam} C(x, y) \le b\,\delta_G(x, y)$ *for all $x, y \in \partial G$.*
(c) $\operatorname{diam} C(x, y) \le b\,|x - y|$ *for all crosscuts $[x, y]$ of $G$.*

*Here the constants $b$ depend only on each other.*

We remark that condition (b) above is equivalent to asking that $G$ satisfy the *crosscut condition*,

$$\min\{\mathrm{diam}(G_1), \mathrm{diam}(G_2)\} \le b\ \mathrm{diam}(\gamma),$$

for each crosscut $\gamma$ of $G$, where $G_1, G_2$ are the components of $G \setminus \gamma$. Similarly, (c) is simply the *straight* crosscut condition,

$$\min\{\mathrm{diam}(G_1), \mathrm{diam}(G_2)\} \le b\,|x-y|,$$

for each crosscut $[x,y]$ of $G$ ($G_1, G_2$ the components of $G \setminus [x,y]$).

That John disks are one-sided quasidisks is made especially clear from the following result due to Näkki and Väisälä [NV91, 9.3].

**2.5. Fact.** *An unbounded Jordan plane curve is a quasicircle if and only if both of its complementary components are John disks. The constants depend only on each other.*

Recall that *quasicircles* and *quasidisks* are the images of circles (or lines) and disks (or half-planes) under a global quasiconformal homeomorphism of the plane. Ahlfors characterized quasicircles as the Jordan plane curves which satisfy a *bounded turning* condition; that is, a Jordan curve $C$ is a quasicircle if and only if there is a constant $b \ge 1$ such that for all $x, y \in C$ we have

$$\mathrm{diam}\, C(x,y) \le b|x-y|.$$

Notice that (b) in Fact 2.4 simply states that $G$ satisfies an *inner bounded turning* condition. Thus we could state Fact 2.5 as: An unbounded Jordan plane curve is bounded turning if and only if both of its complementary components are inner bounded turning.

2.D. **Hyperbolic Geometry.** Recall that the *hyperbolic geodesic lines* in $\mathbf{D}$ are the diameter segments together with the subarcs $S \cap \mathbf{D}$ of the circles $S$ which are orthogonal to the unit circle $\mathbf{S}^1$. The hyperbolic geodesics in a general simply connected proper subdomain of the plane (e.g. in $\mathbf{H}$) are then the images of these under a Riemann map. These geodesics contain the length minimizing arcs for the hyperbolic metric which is defined in the disk via the infinitesimal metric $2(1-|\zeta|^2)^{-1}|d\zeta|$ and then pushed forward to a general hyperbolic domain via any holomorphic covering.

Notice that in a Jordan domain $G$ each geodesic line has well-defined endpoints which lie on $\partial G$. In this case there is always a unique geodesic line $\gamma$ joining any pair of points $x, y \in \partial G$. Then for any $z \in \gamma$ we call $\gamma(z,x)$ a *geodesic ray* from $z$ to $x$.

It is well-known that geodesics in John disks are double cone arcs (see [NV91, 5.2] or [GHM89, 4.1]). For a disk or half-plane this is an immediate consequence of the following often stated fact, which is apparently not recorded in the literature.

**2.6. Lemma.** *Let $\Gamma$ be a geodesic line in $\mathbf{U}$ through the point $\zeta$. Let $\gamma$ be the component of $\Gamma \setminus \{\zeta\}$ having shorter euclidean arclength. Then*

$$\operatorname{dist}(\zeta, \mathbf{T}) \le \ell(\gamma) \le (\pi/2) \operatorname{dist}(\zeta, \mathbf{T}).$$

*Proof.* The left-hand inequality is clear; we verify the right-hand one. This is straightforward for the case when $\mathbf{U}$ is the upper half-plane, so we consider only the unit disk case. We may assume that $\zeta$ is a real number $0 < a < 1$. Then the geodesic lines through $a$ are subarcs of the circles through the points $a, 1/a$ (since these two points are symmetric with respect to the unit circle). Thus we see that the 'short' geodesic rays from $a$ are the images of $[0, e^{it}]$ for $|t| \le \pi/2$ under the correspondence $\zeta = (w + a)/(1 + aw)$. A calculation now gives

$$\ell(\gamma) = \int_{[0,e^{it}]} \frac{(1-a^2)|dw|}{|1+aw|^2} \le \int_0^1 \frac{(1-a^2)\, dr}{1+a^2r^2} = \frac{1-a^2}{a} \arctan(a)$$

and so the desired inequality holds because $(1 + x) \arctan(x) \le (\pi/2)x$ for $0 \le x \le 1$. □

**2.7. Corollary.** *Let $\gamma$ be as above. Put $\gamma_j = \gamma \cap \{z \in \mathbf{U} : 2^{-(j+1)} \le \operatorname{dist}(z, \mathbf{T}) \le 2^{-j}\}$. Choose $i \in \mathbb{Z}$ with $2^{-(i+1)} < \operatorname{dist}(\zeta, \mathbf{T}) \le 2^{-i}$. Then for all $j > i$,*

$$\frac{1}{2^{j+1}} \le \ell(\gamma_j) \le \frac{3}{2^{j+1}};$$

*the righthand inequality also holds for $j = i$.*

*Proof.* We have $\gamma_j = \gamma(\zeta_j, \zeta_{j+1})$ where $\zeta_j$ is the point of intersection of $\gamma$ and (the circle or line) $\operatorname{dist}(z, \mathbf{T}) = 2^{-j}$. Thus $\ell(\gamma_j) \ge |\zeta_{j+1}| - |\zeta_j| = 2^{-(j+1)}$ and the lefthand inequality follows. Letting $\xi = \lim_{k\to\infty} \zeta_k$ be the other endpoint of $\gamma$ we see that for all $j \ge i$,

$$\ell(\gamma(\zeta_j, \xi)) \le 2(\operatorname{dist}(\zeta_j, \mathbf{T})) = 2/2^j \qquad \text{and} \qquad \ell(\gamma(\zeta_{j+1}, \xi)) \ge 1/2^{j+1},$$

so

$$\ell(\gamma_j) = \ell(\gamma(\zeta_j, \xi)) - \ell(\gamma(\zeta_{j+1}, \xi)) \le (2 - 1/2)/2^j = 3/2^{j+1}.$$

□

We make extensive use of the following information regarding the lengths and diameters of geodesics. These are the so-called Gehring-Hayman inequalities which state that geodesics are essentially both the length and diameter minimizers for the inner distances. See [Pom92, Thm4.20,p.88].

**2.8. Fact.** *Let $G$ be a Jordan domain in $\mathbf{R}^2$, let $\gamma$ be a hyperbolic geodesic joining two points of $\bar{G}$ and let $\beta$ be any arc in $G$ joining the endpoints of $\gamma$. Then*

$$\ell(\gamma) \leq k_1\, \ell(\beta) \qquad \textit{and} \qquad \operatorname{diam}(\gamma) \leq k_2 \operatorname{diam}(\beta)$$

*where $k_1, k_2$ are certain absolute constants.*

Here is an especially useful related result [Pom92, Cor4.18,p.87].

**2.9. Fact.** *Let $G$ be a Jordan domain in $\mathbf{R}^2$ and let $A$ be a subarc of $\partial G$. Then for each point $z \in G$, there exist a hyperbolic geodesic $\alpha$ in $G$ joining $z$ to some point of $A$ and satisfying*

$$\ell(\alpha) \leq \operatorname{dist}(z, \partial G) \exp(k/\omega)$$

*where $k$ is an absolute constant and $\omega = \omega(z, A; G)$ is the harmonic measure of $A$ at $z$ relative to $G$.*

We require some notation in order to state the next result. Given a point $\zeta \in \mathbf{U}$, we let $I_\zeta$ be the subarc of $\mathbf{T}$ which is cut off by the hyperbolic geodesic $\Gamma_\zeta$ in $\mathbf{U}$ which goes through $\zeta$ and has $\zeta$ as its Euclidean arclength midpoint. Thus: for $\zeta = \xi + i\eta$ in the upper half-plane $\mathbf{H}$, $I_\zeta = (\xi - \eta, \xi + \eta)$; and for the unit disk $\mathbf{D}$, $I_\zeta = B \cap \mathbf{S}^1$ where $B$ is the disk whose boundary circle $S = \partial B$ has the property that $\Gamma_\zeta = S \cap \mathbf{D}$. The following descriptions for John disks and inner chordarc domains are due to Pommerenke [Pom82, Thm1(ii)], [Pom92, Thm5.2(ii)] and Ghamsari [Gha90, Thm2.19] respectively (at least for the case of bounded domains, but the unbounded case is similar).

**2.10. Fact.** *Let $f : \mathbf{U} \to G$ be conformal with a homeomorphic extension $f : \bar{\mathbf{U}} \to \bar{G}$.*

(a) *$G$ is John if and only if* $\operatorname{diam} f(I_\zeta) \leq b \operatorname{dist}(f(\zeta), \partial G))$.

(b) *$G$ is inner chordarc if and only if* $\ell(f(I_\zeta)) \leq c \operatorname{dist}(f(\zeta), \partial G))$.

*Here both conditions are required to hold for all $\zeta \in \mathbf{U}$, and the associated constants depend only on each other, and, when $G$ is bounded also on* $\operatorname{diam}(G)/\operatorname{dist}(f(0), \partial G)$.

In Section 5 we introduce an $\alpha$-dimensional analog of the Pommerenke-Ghamsari condition (obtained by replacing diameter or length with Hausdorff measure) and use it to characterize $\alpha$-dimensional inner chordarc disks.

## 3. Inner Chordarc Curves

The Ahlfors regular quasicircles are precisely the chordarc curves, also called Lavrentiev or quasismooth curves. These have been studied by David, Jerison and Kenig, Lavrentiev, Pommerenke, Semmes, Tukia, Warschawski, and Zinsmeister; see [Pom92] and the references mentioned therein. The $\alpha$-dimensional chordarc curves were investigated by Ghamsari and the first author [GH98]. Väisälä utilized inner chordarc domains in his solution to the problem of determining which cylindrical domains in $\mathbf{R}^3$ are quasiconformally equivalent to the ball $\mathbf{B}^3$ [Väi89]. Ghamsari further examined these domains in his thesis [Gha90]. In this section we combine these generalizations and inspect the class of $\alpha$-dimensional inner chordarc domains. We confirm the folklore analog that these are precisely the $\alpha$-dimensional inner Ahlfors regular John disks.

3.A. **Inner Chordarc.** We say that a Jordan plane domain $G$ satisfies an *$\alpha$-dimensional inner chordarc* condition if there is a constant $c$ such that for each pair of points $x, y \in C = \partial G$ there is a subarc $A = A(x, y)$ of $C$ joining $x, y$ with

$$c^{-1}\delta_G(x,y)^\alpha \le \mathcal{H}^\alpha(A) \le c\,\delta_G(x,y)^\alpha. \tag{ICA}$$

We abbreviate this by the phrase '$C$ is *c-ICA*$^\alpha$'. We are immediately confronted with the question: Can we take $A = C(x, y)$ or $A = C^\alpha(x, y)$? (Recall that these are the components of $C \setminus \{x, y\}$ with minimal diameter and minimal $\mathcal{H}^\alpha$-measure respectively.) The following is easily verified by mimicking the arguments employed for [GH98, 4.1-4.4] (see Lemma 2.2 also).

**3.1. Fact.** *For a Jordan curve $C = \partial G \subset \mathbf{R}^2$, the following are equivalent:*

(a) (ICA) *holds for some component $A = A(x, y)$ of $C \setminus \{x, y\}$.*
(b) (ICA) *holds for $A = C^\alpha(x, y)$.*
(c) (ICA) *holds for $A = C(x, y)$.*

*Here we require that* (ICA) *holds for all $x, y \in C$; the constant $c$ will vary from* (a) *to* (c)*, but $\alpha$ is fixed. When these conditions hold, $C$ has $\sigma$-finite $\mathcal{H}^\alpha$-measure.*

Next we show that the (HD) condition holds for subarcs of the boundary of $ICA^\alpha$ disks.

**3.2. Lemma.** *Let $G$ be a c-ICA$^\alpha$ disk. Suppose $A$ is any subarc of $C = \partial G$. Then*

$$c^{-1}\,(\operatorname{diam} A)^\alpha \le \mathcal{H}^\alpha(A) \le 2c\,(\operatorname{diam} A)^\alpha\,.$$

*Proof.* For any $x, y \in A$, $\mathcal{H}^\alpha(A) \ge \mathcal{H}^\alpha(C^\alpha(x, y)) \ge c^{-1}\delta_G(x, y)^\alpha \ge c^{-1}|x - y|^\alpha$; this yields the left-hand inequality. Now let $x, y$ be the endpoints of $A$ (which we can assume to be finite) and let $z$ be the

$\mathcal{H}^\alpha$-midpoint of $A$. Then $\mathcal{H}^\alpha(A) = 2\mathcal{H}^\alpha(C^\alpha(x,z)) \le 2c\,\delta_G(x,z)^\alpha \le 2c\,(\operatorname{diam}_\delta A)^\alpha = 2c(\operatorname{diam} A)^\alpha$. □

**3.3. Lemma.** *A c-$ICA^\alpha$ disk is b-John with $b = b(c,\alpha)$.*

*Proof.* We verify the weak inner bounded turning condition (Fact 2.4(c)). Suppose $G$ is $c$-$ICA^\alpha$ with $C = \partial G$ and let $[x,y]$ be a crosscut of $G$. Then for all points $u,v \in C(x,y)$, $C(u,v) \subset C(x,y)$, so

$$c^{-1}\left(|u-v|\right)^\alpha \le c^{-1}\delta_G(u,v)^\alpha \le \mathcal{H}^\alpha(C(x,y)) \le c\,\delta_G(x,y)^\alpha = c\,\left(|x-y|\right)^\alpha$$

and thus $\operatorname{diam} C(x,y) \le c^{2/\alpha}|x-y|$. □

According to Fact 2.3, it now follows that $\lambda_G$ is comparable to $\delta_G$ for an $ICA^\alpha$ domain $G$, so $\lambda_G$ is a continuous distance function on $\bar{G}$. As another application we mention the following. (Alternatively, we could use Lemma 3.2 in conjunction with the fact that a bounded $b$-John disk $G$ contains a point $z$ with $\operatorname{diam}(G) \le 2b\operatorname{dist}(z,\partial G)$.)

**3.4. Corollary.** *Suppose $G$ is a bounded c-$ICA^\alpha$ domain. Then there is a point $z \in G$ such that $\mathcal{H}^\alpha(\partial G) \le c_0 \operatorname{dist}(z,\partial G)^\alpha$ where $c_0 = c_0(c,\alpha)$.*

*Proof.* Select points $x,y \in C = \partial G$ so that both components of $C \setminus \{x,y\}$ have equal measure. Since $G$ is $b$-John, $b = b(c,\alpha)$, there is a double $b$-cone arc $\gamma$ joining $x,y$ in $G$. Let $z$ be the arclength midpoint of $\gamma$. Then

$$\delta_G(x,y) \le \operatorname{diam}(\gamma) \le \ell(\gamma) \le 2b\operatorname{dist}(z,\partial G),$$

and therefore

$$\mathcal{H}^\alpha(C) = 2\mathcal{H}^\alpha(C(x,y)) \le 2c\,\delta_G(x,y)^\alpha \le 2^{1+\alpha}b^\alpha c\operatorname{dist}(z,\partial G)^\alpha.$$

□

3.B. **Inner Ahlfors Regular.** In [GH98, §3] we studied *$\alpha$-dimensional Ahlfors regular* sets $C$ which are defined by the requirement that

$$a^{-1}r^\alpha \le \mathcal{H}^\alpha(C \cap B(x;r)) \le a\,r^\alpha. \tag{AR}$$

for all $x \in C$, all $0 < r \le \operatorname{diam}(C)$ and some constant $a$. We abbreviate this by saying that $C$ is $a$-$AR^\alpha$. The one-dimensional Ahlfors regular Jordan curves were introduced by Ahlfors in his 1935 *Acta* paper on the theory of covering surfaces and are of significant interest in harmonic analysis and singular integral theory because of David's theorem [Dav84] that they are precisely the curves for which the Cauchy integral defines a bounded operator from $L^2$ into $L^2$.

For our purposes we need an inner analog of these, and we declare $G$ to be *$\alpha$-dimensional inner Ahlfors regular*, abbreviated by the phrase '$G$ is $IAR^\alpha$', provided

$$a^{-1}r^\alpha \le \mathcal{H}^\alpha(C \cap B_\delta(x;r)) \le a\, r^\alpha. \tag{IAR}$$

for all $x \in C = \partial G$, all $0 < r \le \operatorname{diam}(C)$ and some constant $a$. Here $B_\delta(x;r)$ denotes the *inner* disk with center $x$ and radius $r$; thus

$$B_\delta(x;r) = \{y \in \bar{G} : \delta_G(x,y) < r\}.$$

It is this $IAR^\alpha$ condition which will prove most useful below. Whereas in general an 'inner condition' is implied by the corresponding 'ordinary' condition, it turns out that $AR^\alpha$ curves need not be $IAR^\alpha$, even for John disks; see Examples 6.2, 6.3. On the other hand, according to Corollary 3.7 an $IAR^\alpha$ John disk is $AR^\alpha$, but see Examples 6.4, 6.5.

Here is one of the crucial 'pieces' in our proof of Theorem A. Recall Lemma 2.2 also.

**3.5. Lemma.** *Suppose $G$ is a Jordan John disk. Then $G$ is $IAR^\alpha$ if and only if* (HD) *holds for each subarc $A \subset C = \partial G$. Here the constants $a$ depend only on each other, $\alpha$ and the John constant.*

*Proof.* We use the Fact 2.4(b) that $G$ is inner bounded turning for some constant $b$. Assume first that $G$ is $a$-$IAR^\alpha$. Let $A$ be a nondegenerate subarc of $C$ and put $r = \operatorname{diam}(A)$. Then for any $w \in A$ we have $A \subset C \cap B_\delta(w;r)$ and so $\mathcal{H}^\alpha(A) \le a\, r^\alpha$. Now select $w \in A$ so that $\operatorname{diam} A(x,w) = \operatorname{diam} A(y,w)$ where $x, y$ are the endpoints of $A$. Since $r = \operatorname{diam}(A) \le \operatorname{diam} A(x,w) + \operatorname{diam} A(y,w)$, $\operatorname{diam} A(x,w) = \operatorname{diam} A(y,w) \ge r/2$ and therefore neither of the subarcs $A(x,w), A(y,w)$ can belong to the disk $B_\delta(w;r/2b)$. Now fix a point $z \in C \cap B_\delta(w;r/2b)$. Then

$$\operatorname{diam} C(z,w) \le b\,\delta_G(z,w) < r/2,$$

so we see that $C(z,w) \subset B_\delta(w;r/2)$. If $z \notin A$, then we would have either $A(x,w) \subset C(z,w)$ or $A(y,w) \subset C(z,w)$, both of which are impossible. We conclude that $C \cap B_\delta(w;r/2b) \subset A$, hence

$$\mathcal{H}^\alpha(A) \ge \mathcal{H}^\alpha\left(C \cap B_\delta(w;r/2b)\right) \ge a^{-1}(r/2b)^\alpha,$$

and so (HD) holds with the constant $a(2b)^\alpha$.

Now suppose (HD) holds and let $w \in C$, $0 < r \le \operatorname{diam}(C)$. If $A$ is the $w$-component of $C \cap B_\delta(w;r)$ then

$$\mathcal{H}^\alpha\left(C \cap B_\delta(w;r)\right) \ge \mathcal{H}^\alpha(A) \ge a(\operatorname{diam} A)^\alpha \ge a\, r^\alpha.$$

On the other hand: if $A$ is the $w$-component of $C \cap B_\delta(w;br)$, then for any $z \in C \cap B_\delta(w;r)$ we have $\operatorname{diam} C(z,w) \le b\,\delta_G(z,w) < br$, so

$C(z,w) \subset B_\delta(w;br)$ which means that $C \cap B_\delta(w;r) \subset A$ and therefore

$$\mathcal{H}^\alpha\left(C \cap B_\delta(w;r)\right) \le \mathcal{H}^\alpha(A) \le a(\operatorname{diam} A)^\alpha \le a\,(2br)^\alpha.$$

Thus $G$ is $a(2b)^\alpha$-$IAR^\alpha$. □

3.C. **Proof of Theorem A.** We demonstrate (a)⇒(b)⇒(c)⇒(a). Assume $G$ is $c$-$ICA^\alpha$. According to Lemma 3.3, $G$ is $b$-John with $b = b(c,\alpha)$. Then Lemma 3.2 in conjunction with Lemma 3.5 permits us to assert that $G$ is $a$-$IAR^\alpha$, $a = a(c,\alpha)$. That (b) $\iff$ (c) is the content of Lemma 3.5. It remains to verify the last implication.

Suppose $G$ is $b$-John and (HD) holds. Fix $x, y \in C = \partial G$ and let $A = C(x,y)$. Then

$$\mathcal{H}^\alpha(C(x,y)) \ge a^{-1}(\operatorname{diam} A)^\alpha \ge a^{-1}\delta_G(x,y)^\alpha$$

and, since $G$ is $b$-John, $\operatorname{diam} C(x,y) \le b\,\delta_G(x,y)$, so

$$\mathcal{H}^\alpha(C(x,y)) \le a(\operatorname{diam} A)^\alpha \le a\,b^\alpha \delta_G(x,y)^\alpha.$$

Thus $G$ is $c$-$ICA^\alpha$ with $c = a\,b^\alpha$. □

3.D. **Ordinary AR and CA.** Here we investigate possible connections between the ordinary and inner Ahlfors regular conditions as well as the associated chordarc conditions. We again mention Example 6.2 which reveals the surprising information that the ordinary Ahlfors regular condition does not imply the inner condition, not even for John disks. Our next result has a number of useful consequences.

**3.6. Proposition.** *Suppose $G \subset \mathbf{R}^2$ is a $b$-John Jordan disk satisfying*

$$\mathcal{H}^\alpha(A) \le a(\operatorname{diam} A)^\alpha \qquad \textit{for all subarcs } A \subset \partial G$$

*for some constant $a$. Then for $w \in C = \partial G$ and $0 < r \le \operatorname{diam}(C)$,*

$$\mathcal{H}^\alpha(C \cap B(w;r)) \le c\,r^\alpha,$$

*where $c = c(a,b,\alpha) = a(2\pi b)^\alpha$.*

*Proof.* Fix $w_0 \in C$ and $0 < r \le \operatorname{diam}(C)$. Let $C_0 = C \cap B_0$, $B_0 = B(w_0;r)$. If $C \subset \bar{B}(w_0;3br)$, then by Lemma 3.2

$$\mathcal{H}^\alpha(C_0) \le \mathcal{H}^\alpha(C) \le a(\operatorname{diam} C)^\alpha \le a(6br)^\alpha.$$

Assume $C \not\subset \bar{B}(w_0;3br)$ and fix a point $w \in C \setminus \bar{B}(w_0;3br)$. Note that any continuum joining $w$ to any point of $\bar{B}_0$ has diameter strictly larger than $(3b-1)r \ge 2br$.

We appeal to Lemma 2.1 to obtain disjoint arcs $\delta_k \subset \partial B_0 \cap G$ with endpoints $x_k, y_k \in C$ such that $C_0 \subset \bigcup \alpha_k$, where $\alpha_k$, $\beta_k$ are the components of $C \setminus \{x_k, y_k\}$ with $w \in \beta_k$. Notice that $\delta_G(x_k,y_k) \le \operatorname{diam}\delta_k \le 2r$. Now since $\beta_k$ joins $w$ to $\bar{B}_0$, $\operatorname{diam}\beta_k > 2br$; on the other hand, according to the John condition (Fact 2.4(b)) $\operatorname{diam} C(x_k, y_k) \le b\delta_G(x_k, y_k) \le 2br$. We conclude that $\alpha_k = C(x_k, y_k)$, so

$$\mathcal{H}^\alpha(\alpha_k) \le a(\operatorname{diam} \alpha_k)^\alpha \le a(b \operatorname{diam} \delta_k)^\alpha \le ab^\alpha \ell(\delta_k)^\alpha$$

and therefore

$$\mathcal{H}^\alpha(C_0) \le \sum_i \mathcal{H}^\alpha(\alpha_i) \le ab^\alpha \sum_k (\ell(\delta_k))^\alpha \le ab^\alpha (\sum_k \ell(\delta_k))^\alpha \le a(2b\pi\, r)^\alpha$$

which yields our assertion. □

Thanks to Theorem A, $ICA^\alpha$ disks satisfy the above hypotheses and we obtain a local version of Corollary 3.4. Another consequence is the following result.

**3.7. Corollary.** *If $G$ is $c$-$ICA^\alpha$, then $\partial G$ is $a$-$AR^\alpha$, $a = a(c, \alpha)$.*

*Proof.* Let $C = \partial G$ with $G$ $ICA^\alpha$. Lemmas 3.2, 3.3 permit the use of Proposition 3.6 which furnishes the upper estimates for $\mathcal{H}^\alpha(C \cap B(x; r))$. The lower estimates follow from $B(x; r) \supset B_\delta(x; r)$ in conjunction with $G$ being $IAR^\alpha$, which holds by Theorem A. □

Thus we see that an $IAR^\alpha$ John disk has an $AR^\alpha$ boundary. Examples 6.4, 6.5 show that the John hypothesis is essential.

A quasidisk $G$ is quasiconvex with respect to length, so $|x - y| \approx \delta_G(x, y) \approx \lambda_G(x, y)$, and therefore every $\alpha$-dimensional chordarc plane curve bounds an $ICA^\alpha$ disk. The converse is false as illustrated by Example 6.1. However, we do have the following analog of Fact 2.5.

**3.8. Corollary.** *Let $C$ be an unbounded Jordan plane curve with complementary components $G$ and $G^*$. Then $C$ is $CA^\alpha$ if and only if both $G$, $G^*$ are $ICA^\alpha$ disks.*

*Proof.* If $C$ is $CA^\alpha$, then it is a quasicircle [GH98, 4.5(a)] so $G$, $G^*$ are both quasiconvex. Assume both $G$, $G^*$ are $ICA^\alpha$ disks. Then by Lemma 3.3 they are both John disks, so by Fact 2.5 $C$ is a quasicircle. Thanks to Corollary 3.7, we also know that $C$ is $AR^\alpha$, so it must also be $CA^\alpha$ by [GH98, 4.6]. □

According to Examples 6.2, 6.3, 6.5, the Ahlfors regular analog of the above fails to hold; in fact neither implication need hold.

The Möbius image of a $c$-$ICA^\alpha$ disk may not be $b$-$ICA^\alpha$ for any given $b$. On the other hand, we do have the following characterization of $CA^\alpha$ curves.

**3.9. Corollary.** *Let $C = \partial G$ for some Jordan plane domain $G$. Then $C$ is $b$-$CA^\alpha$ if and only if every Möbius image $G' \subset \mathbf{R}^2$ of $G$ is $c$-$ICA^\alpha$. Here $b, c$ depend only on each other.*

*Proof.* The necessity is a consequence of the fact that Möbius images of $AR^\alpha$ sets are $AR^\alpha$; see [GH98, 3.1]. The sufficiency is an upshot of [NV91, 9.4] which ensures that such a $G$ must be a quasidisk. $\square$

## 4. Parametrizations

Here we characterize inner chordarc domains $G$ in terms of certain mappings. First we consider parametrizations of the boundary curve $C = \partial G$. Then we extend these to homeomorphisms of $\mathbf{U}$ onto $G$; recall that $\mathbf{T} = \partial\mathbf{U}$ and $\mathbf{U}$ is either the unit disk $\mathbf{D}$ when $G$ is bounded or the upper half-plane $\mathbf{H}$ otherwise. Tukia [Tuk81], and independently Jerison and Kenig [JK82, 1.13, p.227] and also Latfullin [Lat82], established that a Jordan curve in $\mathbf{R}^2$ is chordarc if and only if it is the image of $\mathbf{S}^1$ or $\mathbf{R}$ under a global bilipschitz self-homeomorphism of $\mathbf{R}^2$. Väisälä [Väi87] verified the corresponding result for inner chordarc domains; the necessity half of our proof of Theorem C is based on his ideas.

4.A. **Proof of Theorem B.** Suppose that $G$ is a $c$-$ICA^\alpha$ domain and let $\rho$ be any distance function on $C$ with $\delta_G \le \rho \le \lambda_G$. Let $\varphi : \mathbf{T} \to C$ be a natural parametrization, so $\mathcal{H}^\alpha(\varphi(I)) = h\mathcal{H}^1(I)$ for each arc $I \subset \mathbf{T}$ where $h$ is either $\mathcal{H}^\alpha(C)/2\pi$ if $G$ is bounded or 1 otherwise. Given $x = \varphi(\xi)$ and $y = \varphi(\eta)$ we have $C^\alpha(x,y) = \varphi(\mathbf{T}(\xi,\eta))$, so

$$\delta_G(x,y)^\alpha \approx \mathcal{H}^\alpha(C^\alpha(x,y)) = h\,\mathcal{H}^1(\mathbf{T}(\xi,\eta)) \approx |\xi - \eta|.$$

Since $G$ is a John disk, Fact 2.3 now gives $\rho(x,y)^\alpha \approx |\xi-\eta|$ as desired.

Conversely, assume that $\varphi : (\mathbf{T}, d^{1/\alpha}) \to (C,\rho)$ is $K$-BL. We begin by showing that $G$ is a John disk. To this end, let $x$, $y$ be points of $C$ and choose $\xi,\eta \in \mathbf{T}(\varphi^{-1}(x),\varphi^{-1}(y))$ with $\operatorname{diam} C(x,y) = |\varphi(\xi) - \varphi(\eta)|$. Then

$$\begin{aligned}\operatorname{diam} C(x,y) &\le \rho(\varphi(\xi),\varphi(\eta)) \le K\,|\xi-\eta|^{1/\alpha} \le \\ &\le K\,|\varphi^{-1}(x) - \varphi^{-1}(y)|^{1/\alpha} \le K^2\rho(x,y) \le K^2\lambda_G(x,y).\end{aligned}$$

An appeal to Fact 2.4(c) certifies that $G$ is a John disk, and then Fact 2.3 guarantees that $\delta_G \le \rho \le \lambda_G \le b\,\delta_G$ with $b = b(K)$, so $\varphi : (\mathbf{T}, d^{1/\alpha}) \to (C,\delta_G)$ is $L$-BL where $L = bK$.

Next we certify that

$$(8\pi L)^{-\alpha}\,\mathcal{H}^1(I) \le \mathcal{H}^\alpha(\varphi I) \le L^\alpha\,\mathcal{H}^1(I) \qquad \text{for all arcs } I \subset \mathbf{T}.$$

Since $\mathcal{H}^1(I) \approx \operatorname{diam}(I)$, the above inequalities, in conjunction with $\varphi$ being bilipschitz, will permit us to appeal to Lemma 2.2 to see that $\mathcal{H}^\alpha(A) \approx (\operatorname{diam} A)^\alpha$ for all subarcs of $C$; that $G$ is $c$-$ICA^\alpha$ (with $c = c(K,\alpha)$) will then follow from Theorem A.

Let $I$ be a subarc of $\mathbf{T}$ and put $A = \varphi(I)$. An upper bound for $\mathcal{H}^\alpha(A)$ is easily obtained. Let $r > 0$ and let $\{E_i\}$ be a cover of $I$ by compact sets with diameters at most $r$. Then $\{\varphi(E_i)\}$ is a cover of $A$ and, since

$$\begin{aligned}\operatorname{diam}\varphi(E_i) &= |\varphi(\xi) - \varphi(\eta)| \quad (\text{for some } \xi, \eta \in E_i) \\ &\le \delta_G(\varphi(\xi), \varphi(\eta)) \le L\,|\xi - \eta|^{1/\alpha} \le L(\operatorname{diam} E_i)^{1/\alpha} \le Lr^{1/\alpha},\end{aligned}$$

we see that $\sum(\operatorname{diam}\varphi(E_i))^\alpha \le L^\alpha \sum \operatorname{diam} E_i$ and $\mathcal{H}^\alpha(A) \le L^\alpha\, \mathcal{H}^1(I)$.

Producing a lower bound for $\mathcal{H}^\alpha(A)$ requires more work. First note that when $\mathbf{T} = \mathbf{S}^1$ we may assume $\mathcal{H}^1(I) \le \pi$. Next, choose a subarc $J$ compactly contained in the interior of $I$ with $\mathcal{H}^1(J) \ge (1/2)\mathcal{H}^1(I)$ and let $B = \varphi(J)$. Then $R = \operatorname{dist}(B, C \setminus A) > 0$. Also, for any $z \in B$ and $0 < r < R$, $\bar{B}(z; r) \cap (C \setminus A) = \emptyset$. Let $\varepsilon > 0$, $0 < r < R$ and select a cover of $B$ by disks $B_i = B(z_i; r_i)$ with $z_i \in B$, $r_i \le r$ and so that $\sum r_i^\alpha \le \mathcal{H}^\alpha(B) + \varepsilon$.

Fix a point $w \in C \setminus A$; so $w \notin \bar{B}_i$ for all $i$. Appealing to Lemma 2.1 we obtain disjoint arcs $\delta_{ik} \subset \partial B_i \cap G$ with endpoints $x_{ik}, y_{ik} \in C$ such that $C_i = B_i \cap C \subset \bigcup \alpha_{ik}$, where $\alpha_{ik}$, $\beta_{ik}$ are the components of $C \setminus \{x_{ik}, y_{ik}\}$ with $w \in \beta_{ik}$. Note that for all $i, k$

$$\delta_G(x_{ik}, y_{ik}) \le \operatorname{diam}\delta_{ik} \le 2r_i.$$

Also, since we have disjoint subarcs of $\partial B_i$,

$$\sum_k (\operatorname{diam}\delta_{ik})^\alpha \le (\sum_k \operatorname{diam}\delta_{ik})^\alpha \le (2\pi r_i)^\alpha.$$

Fix $i$ and consider $x_k = x_{ik} = \varphi(\xi_k)$, $y_k = y_{ik} = \varphi(\eta_k)$ say. Since $x_k, y_k \in \partial B_i \cap C$, we must have $x_k, y_k \in A$ and so $\xi_k, \eta_k \in I$. As $I$ is a 'short' subarc of $\mathbf{T}$, $I \supset \mathbf{T}(\xi_k, \eta_k)$. On the other hand, $w \in \beta_k$ but $\varphi^{-1}(w) \notin I$ (since $w \notin A$), so $\varphi^{-1}(\beta_k) = \mathbf{T} \setminus \mathbf{T}(\xi_k, \eta_k)$ and therefore $\varphi^{-1}(\alpha_k) = \mathbf{T}(\xi_k, \eta_k)$. We conclude that

$$\begin{aligned}\operatorname{diam}\varphi^{-1}(\alpha_k) = |\xi_k - \eta_k| \le L^\alpha \delta_G(x_k, y_k)^\alpha \le& \\ \le L^\alpha(\operatorname{diam}\delta_{ik})^\alpha \le L^\alpha(2r_i)^\alpha \le (2Lr)^\alpha&.\end{aligned}$$

Thus $\{\varphi^{-1}(\alpha_{ik})\}$ is a cover of $J$ by sets with diameters at most $(2Lr)^\alpha$; also, since

$$\sum_k \operatorname{diam}\varphi^{-1}(\alpha_{ik}) \le L^\alpha \sum_k (\operatorname{diam}\delta_{ik})^\alpha \le (2\pi L r_i)^\alpha$$

we deduce that $\mathcal{H}^1(I) \le 2\mathcal{H}^1(J) \le 2(2\pi L)^\alpha \mathcal{H}^\alpha(A)$ as desired. $\square$

4.B. **Proof of Theorem C–sufficiency.** Assume $G$ is a Jordan plane domain and there is a homeomorphism $\varphi : \mathbf{U} \to G$ satisfying

$$M^{-1}\operatorname{dist}(\zeta,\mathbf{T})^{1/\alpha-1} \le \underline{\partial}\varphi(\zeta) \quad\text{and}\quad \overline{\partial}\varphi(\zeta) \le M \operatorname{dist}(\zeta,\mathbf{T})^{1/\alpha-1}$$

for all $\zeta \in \mathbf{U}$ for some constant $M \ge 1$. First we demonstrate that

$$K^{-1}\ell(\Gamma)^{1/\alpha} \le \ell(\varphi\Gamma) \le K\ell(\Gamma)^{1/\alpha}$$

for each finite geodesic line $\Gamma$ in $\mathbf{U}$, where $K = c(\alpha)M$. The hypotheses concerning $\underline{\partial}\varphi(\zeta)$, $\overline{\partial}\varphi(\zeta)$ yield

$$M^{-1}\int_\gamma \frac{|d\zeta|}{\operatorname{dist}(\zeta,\mathbf{T})^{1-1/\alpha}} \le \ell(\varphi\gamma) \le M\int_\gamma \frac{|d\zeta|}{\operatorname{dist}(\zeta,\mathbf{T})^{1-1/\alpha}}$$

for any rectifiable arc $\gamma$ in $\mathbf{U}$; cf. [Väi71, 5.3,p.12]. Let $\Gamma$ be any finite geodesic line in $\mathbf{U}$, let $\zeta_0$ be its midpoint, and let $\gamma$ be one of the components of $\Gamma\setminus\{\zeta_0\}$. Choose $i\in\mathbb{Z}$ with $2^{-(i+1)} < \operatorname{dist}(\zeta_0,\mathbf{T}) \le 2^{-i}$ and for $j \ge i$ let $\gamma_j = \gamma \cap \{\zeta\in\mathbf{U} : 2^{-(j+1)} \le \operatorname{dist}(\zeta,\mathbf{T}) \le 2^{-j}\}$.

According to Lemma 2.6 and Corollary 2.7 we have

$$\frac{1}{2^{i+1}} \le \ell(\gamma) \le \frac{4}{2^{i+1}}$$

and also

$$\frac{1}{2^{j+1}} \le \ell(\gamma_j) \le \frac{3}{2^{j+1}} \qquad \text{for all } j > i,$$

with the above righthand inequality holding for $j = i$ as well. Thus

$$\begin{aligned}\ell(\varphi\gamma) \le M\sum_{j\ge i}\int_{\gamma_j}\frac{|d\zeta|}{\operatorname{dist}(\zeta,T)^{1-1/\alpha}} &\le M\sum_{j\ge i}2^{(j+1)(1-1/\alpha)}\ell(\gamma_j)\\ &\le 3M\sum_{j\ge i}2^{-(j+1)/\alpha)} \le K_1\ell(\gamma)^{1/\alpha}\end{aligned}$$

where $K_1 = 3s(\alpha)M$, $s(\alpha) = \sum_0^\infty 2^{-j/\alpha}$. Similarly, $\ell(\varphi\gamma) \ge K_2^{-1}\ell(\gamma)^{1/\alpha}$ where $K_2 = 2M4^{1/\alpha}/s(\alpha)$.

Next, a straightforward argument reveals that $\varphi$ is $M^2$-quasiconformal. Standard boundary behavior facts (e.g., see [Väi71, 17.20,p.61]) now yield a homeomorphic extension of $\varphi$ to $\varphi : \hat{\mathbf{U}} \to \hat{G}$ where $\hat{E}$ denotes the closure of $E \subset \mathbf{R}^2$ in the extended plane. Since $\ell(\varphi\Gamma)^\alpha \approx \ell(\Gamma)$ for each finite geodesic line $\Gamma$ in $\mathbf{U}$, it follows that $\varphi$ must map the point at infinity to itself, and therefore $\varphi : \bar{\mathbf{U}} \to \bar{G}$. Moreover, thanks to Fact 2.8, we find that $\varphi : (\mathbf{T}, d^{1/\alpha}) \to (C,\lambda_G)$ is $K$-BL. Thus, according to Theorem B, $G$ is $c$-$ICA^\alpha$ with $c = c(K,\alpha)$. ☐

4.C. **Proof of Theorem C–necessity.** Assume $G$ is $c$-$ICA^\alpha$. We modify the proof of Väisälä's result [Väi87, 3.4,3.8]. We sketch the argument and refer the interested reader to Väisälä's paper or to [Sul00] for the technical details. We examine only the case where $G$ is unbounded. Let $f : \bar{\mathbf{H}} \to \bar{G}$ be a homeomorphism with $f : \mathbf{H} \to G$ conformal. Next let $g = \psi^{-1} \circ f$ where $\psi : \mathbf{R} \to C$ is an $\mathcal{H}^\alpha$-parametrization chosen so that $g : \mathbf{R} \to \mathbf{R}$ is increasing. Thus for all $u, v \in \mathbf{R}$ we have

$$|gu - gv| = \mathcal{H}^\alpha(C(x, y)) \qquad \text{where } x = fu \text{ and } y = fv.$$

It is not difficult to see that $g$ is quasisymmetric: Let $x, y, z$ be the images of $u, u - t, u + t$ under $f$ where $u \in \mathbf{R}$, $t > 0$. Then $Q = G(y, x, z, \infty)$ is a conformal square and, recalling that the inner distance ratio $s_a/s_b$ is bounded by $k = (e^{2\pi} - 1)/\pi$ [HH88], we deduce that

$$\begin{aligned} &|g(u) - g(u-t)| \geq c^{-1}\lambda_G(x, y)^\alpha \geq c^{-1}k^{-\alpha}\lambda_G(y', z')^\alpha \geq \\ &\geq c^{-2}k^{-\alpha}\mathcal{H}^\alpha(C(y', z')) \geq c^{-2}k^{-\alpha}\mathcal{H}^\alpha(C(x, z)) = c^{-2}k^{-\alpha}|g(u) - g(u+t)| \end{aligned}$$

for certain points $y' \in C(x, y)$, $z' \in C(x, z)$. Interchanging the roles of $y, z$ we conclude that $g$ is $H$-quasiymmetric with $H = k^\alpha c^2$.

Consider the Beurling-Ahlfors extension $g : \mathbf{H} \to \mathbf{H}$ which is $K$-QC and $L$-BL with respect to the hyperbolic metric in $\mathbf{H}$, where $K = L^2$ and $L = L(c, \alpha)$. Define $\varphi = f \circ g^{-1} : \mathbf{H} \to G$. (Notice that $\varphi|_{\mathbf{R}} = \psi$.) Since $f$ is a hyperbolic isometry (being conformal), $\varphi$ is also $L$-BL with respect to the hyperbolic metrics in $\mathbf{H}$ and $G$. Thus

$$k_G(\varphi(\xi), \varphi(\zeta)) \approx h_G(\varphi(\xi), \varphi(\zeta)) \approx h(\xi, \zeta),$$

where $k_G$, $h_G$, $h$ are the quasihyperbolic and hyperbolic metrics in $G$, $\mathbf{H}$ respectively. Dividing by $|\xi - \zeta|$ and letting $\xi \to \zeta$ we find that

$$\partial\varphi(\zeta) \approx \frac{\operatorname{dist}(\varphi\zeta, C)}{\operatorname{dist}(\zeta, \mathbf{R})}.$$

Therefore all that remains is to verify that

$$\operatorname{dist}(\varphi\zeta, C) \approx \operatorname{dist}(\zeta, \mathbf{R})^{1/\alpha},$$

and this is accomplished (as in [Väi87, pp.308,309], cf. [Sul00]) via extremal length estimates. $\square$

## 5. Riemann Mappings

Here we present an $\alpha$-dimensional version of the Pommerenke-Ghamsari mapping condition (see Fact 2.10) and explain how our rendition characterizes inner chordarc disks.

5.A. **Alpha-dimensional Mapping Condition.** Recall our notation explained just prior to Fact 2.10. We say that a Jordan domain $G \subset \mathbf{R}^2$ satisfies an *$\alpha$-dimensional Pommerenke-Ghamsari mapping* condition provided

$$a^{-1}\operatorname{dist}(f\zeta,\partial G)^\alpha \le \mathcal{H}^\alpha(fI_\zeta) \le a \operatorname{dist}(f\zeta,\partial G)^\alpha. \tag{PG}$$

for all $\zeta \in \mathbf{U}$, where $f : \mathbf{U} \to G$ is conformal with a homeomorphic extension $f : \bar{\mathbf{U}} \to \bar{G}$. When $\mathbf{U} = \mathbf{D}$ and $\zeta = 0$ we interpret this to mean that $\mathcal{H}^\alpha(fS) \approx \operatorname{dist}(f0,\partial G)^\alpha$ for any semi-circle $S \subset \mathbf{S}^1$.

We require the following numerical fact.

**5.1. Lemma.** *Let $\alpha > 1$ and $t_1 \ge t_2 \ge \cdots > 0$. Suppose there is a constant $K > 1$ such that,*

$$\sum_{i\ge n} t_i^\alpha \le K\, t_n^\alpha \qquad \textit{for each } n.$$

*Then $\sum_{i\ge 1} t_i \le L\, t_1$, where $L = L(K,\alpha)$.*

*Proof.* Hölder's inequality gives

$$\sum_1^N t_i \le (\sum_1^N t_i^\alpha)^{1/\alpha} N^{1/\beta} \le K^{1/\alpha} N^{1/\beta} t_1$$

so everything is ok for a finite sequence with, e.g., $N \le 2K$ terms (and in this case we can take $L = 2^{1/\beta}K$); here $1/\alpha + 1/\beta = 1$. We assume that there are more than $2K$ terms in the sequence. Write

$$q_i = \frac{t_{i+1}}{t_i} \qquad \text{for } i = 1, 2, \ldots.$$

Then for $j > i$ we have

$$\frac{t_j}{t_i} = q_i q_{i+1} \ldots q_{j-1}.$$

Fix an integer $m$ satisfying $K < m < N$ where $N \le \infty$ is the number of terms $t_i$ (e.g. let $m = [K] + 1$) and put $r = [(K-1)/m]^{1/m\alpha}$. Note that $0 < r < 1$. We claim that in any collection $q_n, q_{n+1}, \ldots, q_{n+m-1}$ of $m$ 'adjacent' $q_i$'s, there exists some $q_j \le r$. Indeed, set

$$p = \min\{q_n, q_{n+1}, \ldots, q_{n+m-1}\} \qquad (= q_j \text{ for some } j)$$

and notice that

$$\begin{aligned} K \ge \sum_{i\ge n} (t_i/t_n)^\alpha &= 1 + q_n^\alpha + q_n^\alpha q_{n+1}^\alpha + \cdots + q_n^\alpha q_{n+1}^\alpha q_{n+m-1}^\alpha + \ldots \\ &\ge 1 + p^\alpha + p^{2\alpha} + \cdots + p^{m\alpha} \ge 1 + mp^{m\alpha}. \end{aligned}$$

Finally, we express $\sum t_i/t_1$ as a sum of products of $q_i$ and consider successive groups of $m$ terms of the latter. In the first such group,

$1+q_1+q_1q_2+\cdots+q_1q_2\ldots q_{m-1}$, we see that each addend is at most 1. The second such group can be written as

$$q_1q_2\ldots q_{m-1}(q_m+q_mq_{m+1}+\cdots+q_m\ldots q_{2m-1});$$

again each of the $m$ addends is at most 1, but also, some factor 'out front' is at most $r$ (and the rest are at most 1). Continuing in this manner we deduce that

$$\begin{aligned}\sum_{i\geq 1} t_i/t_1 &= (1+q_1+q_1q_2+\cdots+q_1q_2\ldots q_{m-1})+\\ &\quad + q_1q_2\ldots q_{m-1}(q_m+q_mq_{m+1}+\cdots+q_m\ldots q_{2m-1})+\cdots\leq\\ &\qquad \leq m+mr+mr^2+\cdots=\frac{m}{1-r}\end{aligned}$$

which provides the asserted result with $L=m/(1-r)$. □

5.B. **Proof of Theorem D.** First, suppose that $G$ is $c$-$ICA^\alpha$. Then $G$ is a John disk by Lemma 3.3, so Pommerenke's mapping condition (see Fact 2.10(a)), in conjunction with Lemma 3.5 and $G$ being $IAR^\alpha$, produces

$$\mathcal{H}^\alpha(fI_\zeta)\approx(\operatorname{diam} fI_\zeta)^\alpha\approx\operatorname{dist}(f\zeta,\partial G)^\alpha$$

as desired, where the constant involved depends only on $c$ and $\alpha$. The converse follows in a similar way as soon we demonstrate that $G$ is a John disk. Thus we assume that $G$ satisfies the Pommerenke-Ghamsari condition (PG) for some conformal mapping $f:\mathbf{U}\to G$. We verify that $G$ satisfies the straight crosscut condition of Fact 2.4(c).

We introduce the following notation. Given points $x,y\in C=\partial G$, we write $\gamma(x,y)$ for the geodesic line joining $x,y$. Thus $\gamma(x,y)=f(\Gamma_\zeta)$ for some $\zeta\in\mathbf{U}$; we let $\Delta(x,y)=\operatorname{dist}(f(\zeta),\partial G)$ and $A(x,y)=f(I_\zeta)$. In terms of this notation (PG) states that $\mathcal{H}^\alpha(A(x,y))\approx\Delta(x,y)^\alpha$. Note that $A(u,v)\subset A(x,y)$ for all $u,v\in A(x,y)$, and also, by Fact 2.8, $\Delta(x,y)\leq k_2\,\delta_G(x,y)$ where $k_2$ is the Gehring-Hayman diameter constant.

Let $[x_0,y_0]$ be a straight crosscut of $G$ and $A=A(x_0,y_0)$. There is no harm in assuming that $x_0+y_0=0$. Choose $v\in A$ so that $d=|v|=\max\{|w|:w\in A\}$. Our goal is to establish that $d\leq c|x_0-y_0|$ for some constant $c$ which depends only on $\alpha$ and the constant $a$ from (PG). Since $\operatorname{diam} C(x_0,y_0)\leq\operatorname{diam} A\leq 2d$, this will give the desired straight crosscut condition. We assume that $d\geq 2|x_0-y_0|$ and put $r=d/(2+k_1\pi)$ where $k_1$ is the Gehring-Hayman length constant (see Fact 2.8).

We order the arc $A$ from $x_0$ to $v$ to $y_0$ and let $X=A(x_0,v)$, $Y=A(y_0,v)$. Since $S(v;r)$ separates $v$ and $[x_0,y_0]$ in $G$, there is a component of $S(v;r)\cap G$ which does this (see [New51, VI.7.1,p.151]); that is, there is a subarc of $S(v;r)$ which is a crosscut of $G$ and separates $v$ and $[x_0,y_0]$ in $G$. Let $x_1,y_1\in A$ be the endpoints of this circular crosscut with $x_1\in X$, $y_1\in Y$; thus $x_0<x_1<v<y_1<y_0$.

We outline the rest of the proof. We will construct numbers $|x_0-y_0|\ge t_1\ge t_2\ge\cdots\ge t_n\to 0$ and points $x_0<x_1<\cdots<x_n\to v$ on $X$ such that

$$r\le\sum_1^\infty t_n\ ,\qquad \Delta(x_j,v)\le a^{2/\alpha}k_2t_j\ ,\qquad t_j\le 2k_3\Delta(x_j,x_{j+1})$$

where $k_2$ is the Gehring-Hayman diameter constant and $k_3$ is an absolute constant obtained from Fact 2.9. From the latter two conditions we will obtain

$$\sum_{j\ge i}t_j^\alpha\le Kt_i^\alpha$$

with $K=K(a,\alpha)=a^4(2k_2k_3)^\alpha$. Lemma 5.1 in conjunction with the first condition above then yields

$$r\le Lt_1\le L|x_0-y_0|\ ,\qquad \text{so}\qquad d\le(2+k_1\pi)L|x_0-y_0|\ ,$$

where $L=L(a,\alpha)$.

Now we explain how to choose $t_i$ and $x_i$. Let $X_1=A(x_0,x_1)$, set $t_1=\delta_G(X_1,Y)$ and let $x_2$ be the last/largest point of $X$ in the $\delta_G$-neighborhood $\bar{B}_\delta(X_1;t_1)$ of $X_1$. In general, let $X_n=A(x_0,x_n)$, set $t_n=\delta_G(X_n,Y)$ and let $x_{n+1}$ be the last/largest point of $X$ in the $\delta_G$-neighborhood $\bar{B}_\delta(X_n;t_n)$ of $X_n$. For later purposes, we select $w_n\in X_n$ with $t_n=\delta_G(w_n,x_{n+1})$; note that we actually have $x_{n-1}<w_n\le x_n$. Also, notice that $\operatorname{diam}A(x_n,x_{n+2})\ge\delta_G(X_n,x_{n+2})>t_n$; since $C$ is a Jordan curve, we must therefore have $t_n\to 0$ and thus also $x_n\to v$.

Let us confirm that $r=|x_1-v|\le\sum_1^\infty t_n$. Let $\beta_n$ be an arc joining $w_n,x_{n+1}$ in $\bar{G}$ with $\operatorname{diam}\beta_n=\delta_G(w_n,x_{n+1})=t_n$. Since $x_{n-1}<w_n\le x_n$, $\beta_n\cap\beta_{n+1}\ne\emptyset$ and so $\cup\beta_n$ contains a continuum joining $x_1=w_1$ to $v$. Thus

$$r=|x_1-v|\le\operatorname{diam}(\cup\beta_n)\le\sum_1^\infty\operatorname{diam}\beta_n=\sum_1^\infty t_n.$$

Next, we establish $\Delta(x_j,v)\le a^{2/\alpha}k_2t_j$. Choose points $w\in X_j$, $u\in Y$ so that $t_j=\delta_G(X_j,Y)=\delta_G(w,u)$. Then $A(w,u)\supset A(x_j,v)$, so

$$\begin{aligned}a^{-1}\Delta(x_j,v)^\alpha\le\mathcal{H}^\alpha(A(x_j,v))&\le\mathcal{H}^\alpha(A(w,u))\le\\&\le a\,\Delta(w,u)^\alpha\le a[k_2\delta_G(w,u)]^\alpha=ak_2^\alpha t_j^\alpha\end{aligned}$$

as desired.

Now we demonstrate that $t_j \le 2k_3\Delta(x_j, x_{j+1})$. Let $\gamma = \gamma(x_j, x_{j+1})$ be the geodesic line joining $x_j, x_{j+1}$ and put $z = f(\zeta)$ where $\zeta$ is the midpoint of $f^{-1}(\gamma)$. Thus $\Delta(x_j, x_{j+1}) = \operatorname{dist}(z, \partial G)$. Let $w$ be the point at infinity if $G$ is unbounded; otherwise, let $w = f(-\zeta/|\zeta|) \in B = C\backslash A$. Let $B_1, B_2$ be the subarcs of $B$ joining $x_0, y_0$ to $w$ respectively, and put $A_1 = B_1 \cup A(x_0, x_j)$, $A_2 = B_2 \cup Y \cup A(x_{j+1}, v)$. According to Fact 2.9, there are geodesic rays $\alpha_k$ joining $z$ to points $u_k$ in $A_k$ for $k = 1, 2$ and satisfying

$$\ell(\alpha_k) \le k_3 \operatorname{dist}(z, \partial G)$$

where $k_3$ is an absolute constant. We show that $\max_{k=1,2} \operatorname{diam} \alpha_k \ge t_j/2$, the upshot of this being our desired inequality. We consider the following three cases: some $u_k \in B$, $u_1 \in X_j$ and $u_2 \in Y$, $u_1 \in X_j$ and $u_2 \in A(x_{j+1}, v)$.

Suppose $u_k \in B$. Since $[x_0, y_0]$ is a crosscut of $G$, it separates $A$ and $B$ in $G$; thus $\alpha_k$ must cross $[x_0, y_0]$. Similarly, $\alpha_k$ must cross the geodesic $\gamma(x_1, y_1)$. Now, thanks to Fact 2.8 and our choice of $r$, $\gamma(x_1, y_1)$ lies outside $B(d/2)$. Thus $\alpha_k$ joins the circle $S(d/2)$ to $[x_0, y_0]$ and so

$$\operatorname{diam} \alpha_k \ge (d - |x_0 - y_0|)/2 \ge |x_0 - y_0|/2 \ge t_j/2.$$

Assume neither $u_1$ nor $u_2$ belongs to $B$. Then $u_1 \in X_j$ and $u_2 \in Y \cup A(x_{j+1}, v)$. Let $\alpha = \alpha_1 \cup \alpha_2$. If $u_2 \in Y$, then $\alpha$ joins $X_j$ to $Y$, so $\operatorname{diam} \alpha \ge \delta_G(X_j, Y) = t_j$. On the other hand, if $u_2 \in A(x_{j+1}, v)$, then $\operatorname{diam} \alpha \ge \delta_G(u_2, X_j) \ge t_j$ since $u_2$ lies past $x_{j+1}$. Thus in both of these cases

$$\max_{k=1,2} \operatorname{diam} \alpha_k \ge \frac{1}{2} \operatorname{diam} \alpha \ge \frac{t_j}{2}.$$

Finally, we have $A(x_i, v) = \cup_{j \ge i} A(x_j, x_{j+1})$, so utilizing the inequalities verified above in conjunction with the (PG) hypothesis we obtain

$$\begin{aligned}(2k_3)^{-\alpha} \sum_{j \ge i} t_j^\alpha &\le \sum_{j \ge i} \Delta(x_j, x_{j+1})^\alpha \le \\ &\le a \sum_{j \ge i} \mathcal{H}^\alpha(A(x_j, x_{j+1})) \le a^2 \Delta(x_i, v)^\alpha \le a^4 k_2^\alpha t_i^\alpha.\end{aligned}$$

As indicated above, Lemma 5.1 now yields

$$r = |x_1 - v| \le \sum_1^\infty t_n \le L t_1 \le L|x_0 - y_0|$$

and thus $d \le (2 + k_1\pi)L|x_0 - y_0|$ as desired. $\square$

## 6. Examples

In this Section we exhibit examples to illustrate our results. According to Corollary 3.8, $\alpha$-dimensional chordarc plane curves are boundaries of $\alpha$-dimensional inner chordarc disks; e.g., the inside of an $\alpha$-dimensional von Koch snowflake curve is an $\alpha$-dimensional inner chordarc disk—see [GH98, §5] for a discussion of these and other curves. The converse is false, as our first example reveals. Here and below we denote by $K^\alpha$ the $\alpha$-dimensional von Koch snowflake arc; thus $K^\alpha$ is the limit (in the Hausdorff metric) of the sequence of piecewise linear Jordan arcs $\{J_n\}$ where $J_0 = [0,1]$, $J_1 = \bigcup_{k=1}^4 I_1^k$ (and $I_1^k = \sigma_k(J_0)$, $\sigma_k$ is a similarity from $J_0$ onto the interval $[a_{k-1}, a_k]$, $a_0 = 0, a_1 = t = (1/4)^{(1/\alpha)}, a_2 = 1/2 + i\sqrt{t - 1/4}, a_3 = 1 - t, a_4 = 1)$, $J_2 = \bigcup_{k=1}^4 I_2^k$ $(I_2^k = \sigma_k(J_1))$, etc.

**6.1. Example.** *For each $\alpha \geq 1$, there is an $\alpha$-dimensional inner chordarc disk $G$ such that $C = \partial G$ is not $\alpha$-dimensional chordarc.*

*Proof.* Consider the points $z_k = (2^{-k}, 4^{-k})$ in $\mathbf{R}^2$, where $k = 0, 1, 2, \dots$; these lie on the graph of a parabola. For $k \geq 1$ we replace each of the segments $[z_k, z_{k-1}]$ by an appropriately scaled copy, $J_k$, of $K^\alpha$; thus $J_k$ is a copy of $r_k K^\alpha$ where $r_k = |z_k - z_{k-1}| \approx 2^{-k}$. We replace the ray $\{(t, 2t-1) : t \geq 1\}$ by $J_0$, a translated and rotated copy of $\cup_{i\geq 1} 3^i K^\alpha$. Then $C = C_+ \cup C_-$ where $C_-$ is the reflection of $C_+ = \cup_{k\geq 0} J_k$.

It is easy to check that $C$ is not a quasicircle, hence not $CA^\alpha$. However, the Jordan region 'to the left' of $C$ is an $ICA^\alpha$ disk. One way to verify this is to show that it is a John disk and that $\mathcal{H}^\alpha(A) \approx (\operatorname{diam} A)^\alpha$ for each subarc $A$ of $C$. □

From Corollary 3.7 we know that the boundary of an $\alpha$-dimensional inner Ahlfors regular John disk satisfies the ordinary $\alpha$-dimensional Ahlfors regular condition. Below we furnish several examples which confirm that this is the only relationship holding between these two regularity conditions. First we present two examples which show that ordinary Ahlfors regularity does not imply the inner condition, even when the domain is John. Then we consider the converse.

Now we construct a John disk which is not $IAR^\alpha$ even though its boundary is $AR^\alpha$.

**6.2. Example.** *Let $1 < \alpha < 2$. There is a Jordan John disk $G \subset \mathbf{R}^2$ with $\partial G$ an $\alpha$-dimensional Ahlfors regular curve, yet $G$ fails to be $\alpha$-dimensional inner Ahlfors regular.*

*Proof.* Let us first describe the principal idea. Below we construct an $AR^\alpha$ Jordan arc $A$ in the closed upper half-plane with the property that $n \in A$ for all $n = 0, 1, 2, \ldots$. We set $C = A \cup B$ where $B$ is the arc obtained by successively joining the points $(0, 0)$, $(0, -1)$, $(1, -1)$, $(2, -1) \ldots$ using copies of $K^\alpha$. Then $C$ is $AR^\alpha$, and the component $G$ of $\mathbf{R}^2 \setminus C$ containing $(-1, 0)$ is a John disk which, as is easily seen, fails to be $IAR^\alpha$. Now we present some details. For convenience we take $\alpha = \log 4/\log 3$; thus $K^\alpha$ is the middle-third von Koch snowflake arc.

We build $A$ by using scaled copies of $K^\alpha$ which we lay on a skeleton obtained from certain squares. Let $Q_n$ be the $3^n \times 3^n$ square whose base is the segment $[0, 3^n]$. We explain how to construct an $AR^\alpha$ arc $F_n$ in $Q_n$ which joins its bottom vertices $(0, 0)$, $(3^n, 0)$. For $n = 0$ we take $F_0 = K^\alpha$. Consider the $n = 1$ case. Here $F_1$ is the union of seven copies of $K^\alpha$; three of these are horizontal ('pointing' up) and join the points $(0, 0)$ to $(1, 0)$, $(1, 2)$ to $(2, 2)$, $(2, 0)$ to $(3, 0)$; the remaining four are vertical and join the points $(1, 0)$ to $(1, 1)$ to $(1, 2)$ and $(2, 2)$ to $(2, 1)$ to $(2, 0)$: we take the first, last of these to 'point' towards each other while the second, third 'point' away from each other.

Now suppose $n = 2$. We obtain $F_2$ by taking two copies of $F_1$, a modified copy of $F_1$, and three copies of $3K^\alpha$. The two copies of $F_1$ are used in the lower $3 \times 3$ corners of $Q_2$ to join $(0, 0)$ to $(3, 0)$ and $(6, 0)$ to $(9, 0)$. We modify $F_1$ by replacing its 'middle horizontal piece' (the subarc joining $(1, 2)$ to $(2, 2)$) with four copies of $K^\alpha$ which join $(1, 2)$ to $(1, 3)$ to $(0, 3)$ and $(2, 2)$ to $(2, 3)$ to $(3, 3)$ (the two vertical arcs 'point' away from each other and the two horizontal ones 'point' up). We place this modified copy onto the lower middle $3 \times 3$ subsquare of $Q_2$. To complete $F_2$ we add three copies of $3K^\alpha$ which join the points $(3, 3)$ to $(3, 6)$ to $(6, 6)$ to $(6, 3)$ (again, the vertical arcs 'point' away from each other and the horizontal one 'points' up).

In general we construct $F_{n+1}$ by using two copies of $F_n$ in each of the lower $3^n \times 3^n$ corners of $Q_{n+1}$ and a modified copy of $F_n$ in the lower middle $3^n \times 3^n$ subsquare to which we add three copies of $3^n K^\alpha$ which join the points $(3^n, 3^n)$ to $(3^n, 2 \cdot 3^n)$ to $(2 \cdot 3^n, 2 \cdot 3^n)$ to $(2 \cdot 3^n, 3^n)$ (the vertical arcs 'point' away from each other and the horizontal one 'points' up). We find, for example, that

$$2(3^n)^\alpha \leq \mathcal{H}^\alpha(F_n) \leq 6(3^n)^\alpha$$

and the same inequalities hold for the modified $F_n$ arcs. It is straightforward to verify that each fractal arc $F_n$ is $a$-$AR^\alpha$ for some absolute constant $a$.

We place the squares $Q_n$ adjacent to one another along the horizontal axis; so $Q_n$ now has base $[(x_n, 0), (x_{n+1}, 0)]$ where $x_0 = 0$ and $x_{n+1} = x_n + 3^n = (3^{n+1} - 1)/2$. We set $A = \cup_{n \geq 0} F_n$, where now $F_n$ has been moved along with $Q_n$ (so $F_n$ has endpoints $(x_n, 0)$ and $(x_{n+1}, 0)$). We claim that $A$ is $AR^\alpha$. One way to see this is to first check the points $x_n$; for $0 < r \le 3^n$ all is ok since $F_{n-1}$ and $F_n$ are $AR^\alpha$, and for $r > 3^n$ we find that $r \approx 3^m$ and $A(x_{n-1}, x_{m-1}) \subset A \subset A(x_0, x_m)$, where $x_m$ is the last vertex in the square $Q(x_n; r)$, so all is ok. To establish the $AR^\alpha$ condition for a general point $x$ we employ a closest point $x_n$.

Having constructed $A$, we now set $C = A \cup B$, where $B$ is described above, and we let $G$ be the complementary component 'to the left' of $C$. Employing the straight crosscut condition (Fact 2.4(c)) it is not difficult to see that $G$ is a John disk. Consider a point $w = (m, -1) \in B$ for some integer $m \ge 1$. We see that $\mathcal{H}^\alpha(C \cap B_\delta(w; m)) = 2m$, and since $m^{1-\alpha} \to 0$ as $m \to \infty$ we easily conclude that $G$ is not $IAR^\alpha$. It remains to explain why $C$ is $AR^\alpha$. It is not difficult to see that everything is ok at points $w \in A$; the lower estimate for $\mathcal{H}^\alpha(C \cap B(x; r))$ clearly still holds since $C \supset A$, and the upper estimate also holds because $C \cap B(x; r)$ does not contain much more measure than $A \cap B(x; r)$. Consider a point $w \in B$. For small $r > 0$ everything is ok because the middle third von Koch snowflake is $CA^\alpha$. For large $r$ we let $n \ge 0$ be an integer closest to $w$. Then we have $Q(n; r/2) \subset Q(w; r) \subset Q(n; 2r)$ and again the measure estimates follow. □

Next we construct an unbounded $AR^\alpha$ curve such that neither complementary component is $IAR^\alpha$.

**6.3. Example.** *Let $1 < \alpha < 2$. There is an unbounded $\alpha$-dimensional Ahlfors regular Jordan plane curve both of whose complementary components are not $\alpha$-dimensional inner Ahlfors regular.*

*Proof.* As above we consider the case $\alpha = \log 4/\log 3$. We utilize the fractal arcs $F_n$ constructed in the previous example. We let $E_n = A((x_{n+1}, -1), (x_{n+1}, 0))$ where $A$ and $x_n$ are as in Example 6.2; thus $E_n$ is a long 'horizontal' arc which consists of $\cup_0^n F_i$ together with a union of many copies of $K^\alpha$. We use 'vertical' copies $E_n^+$, $E_n^-$ of $E_n$, one 'pointing' up and one down. These are joined together by a 'monster' middle-third von Koch snowflake curve

$$M = K \cup (-K) \qquad \text{where} \qquad K = \bigcup_{i=0}^{\infty} 3^i K^\alpha.$$

Let $(t_n, 0)$ denote the endpoint of some arc $3^i K^\alpha$ chosen so that $t_{n+1} >> t_n$. We replace the subarcs of $M$ joining $(t_n, 0)$ to $(t_n + 1, 0)$, and $(-t_n, 0)$ to $(-t_n - 1, 0)$, by the arcs $E_n^-$, and $E_n^+$ respectively, ('vertical' copies of $E_n$ 'pointing' downward, upward resp.); here we take $E_n^-$ and $E_n^+$ to be symmetric with respect to the origin, so the resulting Jordan curve $C$ is also symmetric. Similar reasoning as in Example 6.2

now certifies that $C$ is $AR^\alpha$, but neither of its complementary components is $IAR^\alpha$. $\square$

Our last examples demonstrate that the $IAR^\alpha$ condition does not imply the $AR^\alpha$ condition, even if both complementary components are $IAR^\alpha$. This completes the picture regarding ordinary versus inner Ahlfors regularity.

**6.4. Example.** *Let $1 \le \alpha < 2$. There is an unbounded Jordan plane curve which has an $\alpha$-dimensional inner Ahlfors regular complementary component, but the curve itself fails to be $\alpha$-dimensional Ahlfors regular.*

*Proof.* Our example is modeled on the curve

$$C = (-\infty, 0] \cup \bigcup_{n=0}^{\infty} (T_n \cup [2n+1, 2n+2])$$

where $T_n = R_n \setminus (2n, 2n+1)$ and $R_n$ is a rectangle with base $[2n, 2n+1]$ and height $2^n$. We leave it to the reader to verify that $C$ is not Ahlfors regular whereas the Jordan domain 'above' $C$ is inner Ahlfors regular.

We explain our construction for the dimension $\alpha = \log 4/\log 3$. First we describe certain 'level' curves $L_n$ which we use to build 'fractal shadow' arcs $\Lambda_n$. These in turn are used to define 'zig-zag' arcs $\Gamma_n$, and then $C$ is obtained by pasting together translates of the $\Gamma_n$'s.

We denote by $\mathcal{K}{=}K^\alpha$ the middle-third von Koch snowflake arc; recall that $\mathcal{K}$ is the limit (in the Hausdorff metric) of a sequence of piecewise linear Jordan arcs $\{J_n\}$ where $J_0 = [0,1]$, $J_1$ is obtained by replacing $[1/3, 2/3]$ (the 'middle-third' of $J_0$) with two edges of an equilateral triangle 'pointing up' and having base $[1/3, 2/3]$, and so forth. Thus, e.g., $J_1$ consists of four cells (line segments) each of size one-third having endpoints $a_0 = 0$, $a_1 = 1/3$, $a_2 = 1/2 + i/2\sqrt{3}$, $a_3 = 2/3$, $a_4 = 1$. In general, $J_n$ consists of $4^n$ cells each of size $s_n = 3^{-n}$.

In preparation for later use, we put

$$\mathcal{K}^0 = \mathcal{K}, \quad \text{and for } m \ge 1, \quad \mathcal{K}^m = 3^m\mathcal{K} + (1 + 3 + \cdots + 3^{m-1});$$

thus $\mathcal{K}^m$ is a copy of $\mathcal{K}$ which has been magnified by a factor of $3^m$ and then translated horizontally to the right. Notice that $\mathcal{M} = \bigcup_{m=0}^{\infty} \mathcal{K}^m$ is a 'monster' snowflake arc joining the origin to the point at infinity obtained by placing larger and larger copies of $\mathcal{K}$ adjacent to each other along the positive real axis.

We associate with each arc $J_n$ a sort of 'level curve' $L_n$ described as follows. Roughly speaking, we begin by taking (most of) the line segments in that part of the level curve $\{\mathrm{dist}(z, J_n) = r_n = s_n/\sqrt{3}\}$

which lies *below* $J_n$ (so we delete any circular arcs and everything which is not below $J_n$) and then we join the separate pieces with more line segments.

Now we give a precise definition of $L_n$. First, $L_1$ consists of three horizontal cells (line segments) of size $s_1$ which lie directly below $J_1$ at a 'distance' $r_1$; thus $L_1 = [-ir_1, 1 - ir_1]$. Next we look at $J_2$; recall that it consists of four contracted (by $1/3$) copies of $J_1$. Consider the corresponding copies of $L_1$; we leave the leftmost and rightmost copies alone. A careful inspection of the two middle copies reveals that they intersect; in fact, we can remove one cell from each of these copies of $L_1$ and then the two remaining copies will just touch at a point directly below the 'upward pointing' vertex $a_2$ of $J_2$. At this stage we now have three 'pieces': The first and last are just the leftmost and rightmost scaled copies of $L_1$ (each of these consists of three cells of size $s_2$ which lie at a 'distance' $r_2$ below $J_2$). The middle 'piece' consists of four cells each of size $s_2$ at a 'distance' $r_2$ below the middle of $J_2$; this middle 'piece' can also be described as the top two edges of an equilateral triangle. We call these ten segments the *ordinary* cells in $L_2$. To complete the construction of $L_2$ we use two *short* cells to join together the three separate pieces described above; these short cells are located near the 'downward pointing' vertices $a_1$, $a_3$ of $J_2$. We easily see that these two cells have size (length) $r_2$. The reader might benefit from a picture; note especially how $L_2$ compares with that part of the level curve $\{z : \mathrm{dist}(z, J_2) = r_2\}$ which lies below $J_2$.

We construct $L_n$ by iterating the above procedure. Recall that $J_{n+1}$ consists of four contracted copies of $J_n$. As before, we take the associated copies of $L_n$, leave the two end copies alone, delete one ordinary cell from each of the two middle copies (so that they touch just below the upward pointing vertex $a_2$), and add two short cells (just below the downward pointing vertices $a_1, a_3$) to connect everything together. We see that $L_{n+1}$ consists of $\nu_{n+1} = 4\nu_n - 2$ ordinary cells of size $s_{n+1}$ and $\mu_{n+1} = 4\mu_n + 2$ short cells of size $r_{n+1}$. Since $\nu_1 = 3$, $\mu_1 = 0$, $\nu_2 = 10$, $\mu_2 = 2$ we find that

$$\nu_n = (7 \cdot 4^{n-1} + 2)/3 \quad \text{and} \quad \mu_n = (2 \cdot 4^{n-1} - 2)/3.$$

Again we encourage the reader to think of $L_n$ as that part of the level curve $\{z : \mathrm{dist}(z, J_n) = r_n\}$ which lies below $J_n$.

We are ready to describe the *fractal shadow* arcs $\Lambda_n$. We simply replace each cell of $L_n$ with an appropriately transformed 'upward pointing' copy of $\mathcal{K}$. After a short calculation we obtain

$$\mathcal{H}^\alpha(\Lambda_n) = \nu_n s_n^\alpha + \mu_n r_n^\alpha = (2 + 4^{-n})/3.$$

A geometric calculation reveals that $\Lambda_n \cap \Lambda_{n+2} = \emptyset$ for all $n$. Thus we see that these fractal shadows can be used in place of a large number of parallel line segments packed closely together.

Next we use the arcs $\Lambda_n$ to construct 'zig-zag' arcs $\Gamma_i$ which go back and forth underneath $\cup_0^i \mathcal{K}^m$ and contain more and more closer and closer copies of the fractal shadows. This property will 'destroy' the $AR^\alpha$ condition. We eventually obtain our desired curve $C$ by pasting together translated copies of the $\Gamma_i$ arcs.

Recall that $\mathcal{K}^m = 3^m\mathcal{K} + (1 + 3 + \cdots + 3^{m-1})$; associated with each such scaled/translated copy of $\mathcal{K}$ we also have scaled/translated copies of the arcs $J_n$, $L_n$, $\Lambda_n$. We write $\Lambda_n^m$ for the copy of $\Lambda_n$ which sits below $\mathcal{K}^m$. We see that, since $\Lambda_n$ is 'at a distance' $r_n$ from $\mathcal{K}$, $\Lambda_n^m$ is 'at a distance' $3^m r_n = r_{n-m}$ from $\mathcal{K}^m$ (provided $n > m$).

We start with $\Gamma_0 = A_0^0 = \Lambda_2$. Next,

$$\Gamma_1 = \Gamma_0 \cup E_1 \cup A_1^0 \cup F_1 \cup B_1^0 \cup A_0^1.$$

Here $E_1$ and $F_1$ are 'connecting fractals' obtained by replacing line segments with 'outward pointing' contracted copies of $\mathcal{K}$, $A_1^0 = \Lambda_4$, $B_1^0 = \Lambda_6$, and $A_0^1$ is an appropriate copy (magnified and translated to lie below $\mathcal{K}^1$) of some fractal shadow chosen so that it will join up with $B_1^0$. In fact, since $B_1^0$ is 'at distance' $r_6$ from $\mathcal{K}^0$, we want $A_0^1$ to also be 'at distance' $r_6 = 3^1 r_7$ from $\mathcal{K}^1$ and thus we find that $A_0^1 = \Lambda_7^1$. We see that $\Gamma_1$ goes left to right along $\Gamma_0$ below $\mathcal{K}^0$, up $E_1$, back along $A_1^0$, up $F_1$, again left to right along $B_1^0$, and continues left to right below $\mathcal{K}^1$ along $A_0^1$. Moreover, $\Gamma_1$ is a Jordan arc because $\Lambda_2, \Lambda_4, \Lambda_6$ do not touch each other.

We iterate this construction by letting

$$\Gamma_2 = \Gamma_1 \cup E_2 \cup A_1^1 \cup A_2^0 \cup F_2 \cup B_2^0 \cup B_1^1 \cup A_0^2$$

where $E_2, F_2$ are 'connecting fractals' and, e.g., $A_2^0 = \Lambda_8$, $B_2^0 = \Lambda_{10}$, and $A_0^2 = \Lambda_{12}^2$. In general,

$$\Gamma_n = \Gamma_{n-1} \cup E_n \cup \bigcup_{i=1}^{n} A_i^{n-i} \cup F_n \cup \bigcup_{i=1}^{n} B_i^{n-i} \cup A_0^n.$$

Here $E_n, F_n$ are 'connecting fractals' and $A_{n-j}^j$, $B_{n-j}^j$ are appropriate fractal shadow arcs $\Lambda_k^j$ lying below $\mathcal{K}^j$; for example,

$$A_n^0 = \Lambda_{4n} \quad , \quad B_n^0 = \Lambda_{4n+2} \quad , \quad A_0^n = \Lambda_{5n+2}^n.$$

Once again, $\Gamma_n$ is a Jordan arc which 'zig-zags' back and forth underneath $\cup_0^n \mathcal{K}^i$ having more and more (in fact, $2n + 1$) fractal shadows directly below $\mathcal{K}^0$.

Now we vertically translate $\Gamma_n$ up by $r_2 = s_2/\sqrt{3} = 1/9\sqrt{3}$ so that its initial point is now at the origin. Next let $z_n = x_n + iy_n$ be the (new) terminal point of $\Gamma_n$; so $x_n = 1+3+\cdots+3^n = (3^{n+1}-1)/2$ and $y_n = r_2 - r_{4n+2}$. We require the following properties of $\Gamma_n$.

$$\begin{gathered} 3^n \le \operatorname{diam}(\Gamma_n) = |z_n| \le \frac{1}{2}3^{n+1}, \\ \frac{2}{3}4^n \le \mathcal{H}^\alpha(\Gamma_n) \le 2\cdot 4^n, \\ \mathcal{H}^\alpha(A) \approx (\operatorname{diam} A)^\alpha \quad \text{for all subarcs } A \subset \Gamma_n. \end{gathered}$$

It is straightforward to estimate the measure of $\Gamma_n$. An easy, but tedious, case by case analysis corroborates the $\mathcal{H}^\alpha/\operatorname{diam}$ condition; here $\approx$ means that the ratio of the two quantities is bounded above and below by absolute constants. It is this last property which is most useful in establishing the inner Ahlfors regularity of our curve. As a warm-up exercise the reader can check that we have

$$\mathcal{H}^\alpha(\Gamma_n \cap B_\delta(z;r)) \approx r^\alpha \quad \text{for all } 0 < r < \operatorname{diam}(\Gamma_n),\ z \in \Gamma_n$$

where we are (temporarily) thinking of $\delta$ as the inner distance in the region lying below the curve $(-\infty, 0] \cup \Gamma_n \cup [z_n, +\infty)$ formed by the negative real axis followed by $\Gamma_n$ followed by a horizontal ray from $z_n$ parallel to the positive real axis.

We are finally ready to construct $C$. We put

$$C = S \cup \bigcup_{n=0}^{\infty} \Sigma_n$$

where $S = \{-z : z \in \mathcal{M}\}$, $\Sigma_0 = \Gamma_0$, and $\Sigma_n$ is just $\Gamma_n$ translated to have initial point $\zeta_n$ equal to the terminal point of $\Sigma_{n-1}$ and terminal point $\zeta_{n+1} = \zeta_n + z_n$. We claim that $C$ fails to be $\alpha$-dimensional Ahlfors regular, however, the Jordan domain $G$ which lies below $C$ is $\alpha$-dimensional inner Ahlfors regular.

It is easy to see that $C$ is not $AR^\alpha$; indeed $C \cap B(\zeta_n; 2)$ contains (a copy of) that part of $\Gamma_n$ below $\mathcal{K}^0$, so it contains (copies of) the arcs $A_0^0, A_1^0, \ldots A_n^0, B_1^0, B_2^0, \ldots B_n^0$ which are just $\Lambda_2, \Lambda_4, \ldots, \Lambda_{4n+2}$. Thus $C \cap B(\zeta_n; 2)$ contains $2n+1$ subarcs each having measure at least $2/3$, so $C$ does not satisfy the $AR^\alpha$ condition.

We leave it to the reader to demonstrate that $G$ is $IAR^\alpha$, but remark that the $\mathcal{H}^\alpha/\operatorname{diam}$ condition

$$\mathcal{H}^\alpha(A) \approx (\operatorname{diam} A)^\alpha \quad \text{for all subarcs } A \subset C$$

is a useful fact. Indeed, by looking at the $z$-component of $C \cap B_\delta(z;r)$ we immediately obtain, via this property, that $\mathcal{H}^\alpha(C \cap B_\delta(z;r))$ is bounded below by an absolute constant times $r^\alpha$.

□

A more complicated construction, using the fractal shadow arcs $\Lambda_n$ defined just above, yields our final example.

**6.5. Example.** *Let $1 \le \alpha < 2$. There is an unbounded Jordan plane curve both of whose complementary components are $\alpha$-dimensional inner Ahlfors regular, but the curve itself fails to be $\alpha$-dimensional Ahlfors regular.*

Verification of the details is left for the industrious reader, however, we mention that the curve

$$C = (-\infty, 0] \cup \bigcup_{n=0}^{\infty} T_n$$

has the desired properties for the dimension $\alpha = 1$, where $T_n = (R_n \setminus (2n, 2n+1)) \cup (S_n \setminus (2n+1, 2n+2))$, $R_n$ is a rectangle with base $[2n, 2n+1]$ and height $2^n$, and $S_n$ is a rectangle with top $[2n+1, 2n+2]$ and height $2^n$ (so $R_n$ 'points up' while $S_n$ 'points down').

Recall that an unbounded Jordan plane curve is chordarc if and only if both of its complementary components are inner chordarc disks; the $\alpha$-dimensional analog of this is Corollary 3.8. A similar phenomena holds for the bounded turning condition; see Fact 2.5. The above examples demonstrate that both implications of the analog for Ahlfors regularity fail to hold: When the dimension $\alpha > 1$, the complementary components of an $AR^\alpha$ curve are not necessarily $IAR^\alpha$. (This does hold when $\alpha = 1$.) Conversely, if both components of an unbounded Jordan curve are $IAR^\alpha$, the curve need not be $AR^\alpha$ (and here $1 \le \alpha < 2$).

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