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RNC WORKSHOP

A Möbius invariant metric for regions on the Riemann sphere

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§1 Introduction

Kulkarni and Pinkall [5] introduced a canonical metric relative to a Möbius structure on an n-dimensional manifold. Henceforth, we call this the K-P metric. For the most part they investigated the metric in this general context although they briefly considered the special case of regions in $\mathbb{R}^n$. The purpose of this paper is three-fold. First, we give a different definition of the K-P metric and present new derivations of the basic properties of the metric in the context of regions on the Riemann sphere and from the perspective of classical geometric function theory. We hope this new viewpoint will make this interesting metric better known among classical function theorists. Second, in this particular context more precise information is available than in the general situation investigated by Kulkarni and Pinkall; we obtain a number of new results. Finally, this paper sets the stage for subsequent research in which the authors will provide more specific connections between this metric and other metrics on subregions of the sphere.

A summary of basic information concerning conformal metrics is given in §2. In particular, the hyperbolic metric $\lambda_\Omega(z)|dz|$ on a hyperbolic region Ω is discussed there. We define the K-P metric in terms of the hyperbolic metric of disks in the Riemann sphere; this is not the original definition given by Kulkarni and Pinkall. A disk on the sphere is either an open disk in the plane, an open half-plane or the complement of a closed disk. The density of the K-P metric, denoted by $\mu_\Omega(z)|dz|$ for a region Ω in $\mathbb{P}$, is defined as follows. For $z \in \Omega$ set

$$\mu_\Omega(z) = \inf\left\{\lambda_D(z) : z \in D \subseteq \Omega, D \text{ is a disk in } \mathbb{P}\right\}.$$

For simplicity we shall not use local coordinates; equivalently, we assume the region lies in the complex plane $\mathbb{C}$. The reader can make the necessary modifications if ∞ lies in the region. Henceforth, we say that D is an *extremal disk* (relative to the K-P metric) for $a \in \Omega$ to mean $a \in D \subseteq \Omega$, D is a disk in $\mathbb{P}$ and $\mu_\Omega(a) = \lambda_D(a)$. For the Riemann sphere or a once-punctured sphere the K-P metric vanishes identically since the density of the hyperbolic metric of a Euclidean disk tends to zero as the radius of the disk tends to infinity. The metric $\mu_\Omega(z)|dz|$ is nontrivial provided $\mathbb{P} \setminus \Omega$ contains at least two points. The fact that Möbius transformations map disks to disks together with the Möbius invariance of the hyperbolic metric gives the Möbius invariance of the K-P metric.

In certain respects, the K-P metric can be regarded as a Möbius invariant analog of the quasihyperbolic metric. The quasihyperbolic metric for a plane region Ω is $|dz|/\delta_\Omega(z)$, where $\delta_\Omega(z) = \operatorname{dist}(z, \partial\Omega)$ is the Euclidean distance to the boundary of Ω. The quasihyperbolic metric coincides with the hyperbolic metric on half-planes and is invariant under similarity transformations $z \mapsto az + b$, $a \neq 0$. It was introduced by Gehring and Palka [2] and has proven fruitful in geometric function theory. In particular, useful bounds on the hyperbolic metric $\lambda_\Omega(z)|dz|$ can be given in terms of the quasihyperbolic metric. A hyperbolic plane region is called uniformly perfect if and only if the quasihyperbolic and hyperbolic metrics on the region are equivalent. Since $\lambda_\Omega \leq 2/\delta_\Omega$ holds for any hyperbolic plane region, Ω is uniformly perfect precisely when

$$0 < c(\Omega) = \inf\{\delta_\Omega(z)\lambda_\Omega(z) : z \in \Omega\}.$$

We show the K-P metric has the following properties. Many were obtained by Kulkarni and Pinkall [5]. Most are analogs of results for the hyperbolic metric.

I. The infimum in the definition of the K-P metric can be replaced by a minimum and the extremal disk is unique for regions with at least two boundary points;

II. The K-P metric is invariant under all Möbius transformations;

III. The K-P metric is monotonic with respect to the region: $\Omega \subseteq \Delta$ imples $\mu_\Delta \leq \mu_\Omega$ on Ω;

IV. The K-P metric is complete for regions with at least two boundary points;

V. The K-P metric varies continuously with the region in the sense of kernel convergence;

VI. The K-P metric is quasi-invariant under conformal mappings: if $f : \Omega \to \Delta$ is a conformal mapping, then $\frac{1}{2}\mu_\Omega(z) \leq \mu_\Delta(f(z))|f'(z)| \leq 2\mu_\Omega(z)$. These bounds are best possible;

VII. The K-P and quasihyperbolic metrics are equivalent on any proper subregion of the plane; in fact, $1/\delta_\Omega \leq \mu_\Omega \leq 2/\delta_\Omega$;

VIII. $\lambda_\Omega \le \mu_\Omega$, and $\mu_\Omega(z)|dz|$ is comparable to $\lambda_\Omega(z)|dz|$ if and only if Ω is uniformly perfect. For Ω simply connected, the exact bounds $\frac{1}{2}\mu_\Omega \le \lambda_\Omega \le \mu_\Omega$ hold.

Roughly speaking, the K-P metric provides a better approximation to the hyperbolic metric than the quasihyperbolic metric does. For instance, the known bounds $1/(2\delta_\Omega) \le \lambda_\Omega \le 2/\delta_\Omega$ for simply connected regions Ω follow from VII and VIII. Kulkarni and Pinkall [5] also proved that $\mu^2(z)|dz|^2$ is a Riemannian metric of class $C^{1,1}$, that is, with partial derivatives that satisfy a Lipschitz condition.

§2 Preliminaries

We begin with a discussion of conformal metrics on a region Ω in the Riemann sphere $\mathbb{P}$. We should work in terms of local coordinates so the definitions permit ∞ to lie in the region. For simplicity we shall not use local coordinates; equivalently, we assume the region lies in the complex plane $\mathbb{C}$. The reader should make the necessary modifications when ∞ lies in the region.

A conformal metric on Ω is $\sigma(z)|dz|$, where σ is a nonnegative Borel measurable function on Ω. Typically, σ will be positive and continuous, or even smooth. A conformal metric induces a pseudodistance function on Ω given by

$$d_\sigma(a,b) = \inf \int_\gamma \sigma(z)|dz|,$$

where the infimum is taken over all locally rectifiable paths γ in Ω that join a and b. If σ is positive and continuous, then d_σ is a distance function on Ω that is compatible with the topology on Ω as a subset of the Riemann sphere. In this situation d_σ induces the Euclidean topology on a plane region. A path δ in Ω joining a and b is a σ-geodesic if

$$d_\sigma(a,b) = \int_\delta \sigma(z)|dz|.$$

The conformal metric $\sigma(z)|dz|$ is complete if the associated distance function d_σ is complete in the sense that all Cauchy sequences are convergent. If σ is of class C^2, then the Gauss curvature of $\sigma(z)|dz|$ is

$$K(z,\sigma) = -\frac{\Delta \log \sigma(z)}{\sigma^2(z)},$$

where $z = x + iy$ and

$$\Delta = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} = 4\frac{\partial^2}{\partial z \partial \bar{z}}$$

denotes the usual Laplacian. If $f : \Omega \to \Delta$ is a holomorphic function and $\tau(z)|dz|$ a conformal metric on Δ, then the pull-back of $\tau(z)|dz|$ by f is the conformal metric on Ω given by

$$f^*(\tau(z)|dz|) = \tau(f(z))|f'(z)||dz|.$$

Similarly, the pull-back by a conjugate holomorphic function $f : \Omega \to \Delta$ is given by

$$f^*(\tau(z)|dz|) = \tau(f(z)) \left| \frac{\partial f}{\partial \bar{z}}(z) \right| |dz|.$$

We sometimes abuse notation and simply write $f^*(\tau)$ for the pull-back. If τ is of class C^2 on Δ and f is locally injective, then the pull-back is again of class C^2 and

$$K(z, f^*(\tau)) = K(f(z), \tau).$$

In words, the Gaussian curvature of a smooth metric is invariant under pull-back by a locally injective (conjugate) holomorphic function.

The logarithmic, or quasihyperbolic, metric on the punctured plane $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ is $|dz|/|z|$. The function $\exp : \mathbb{C} \to \mathbb{C}^*$ is a holomorphic covering and $\exp^*(|dz|/|z|) = |dz|$, the Euclidean metric on $\mathbb{C}$. Since the Euclidean metric has zero curvature and curvature is invariant under a locally injective holomorphic function, the logarithmic metric has zero curvature. The completeness of the Euclidean metric on $\mathbb{C}$ implies the logarithmic metric is also complete.

We recall the definition of the hyperbolic metric. The hyperbolic metric on the unit disk $\mathbb{D} = \{z : |z| < 1\}$ is

$$\lambda_{\mathbb{D}}(z)|dz| = \frac{2|dz|}{1 - |z|^2};$$

it is normalized to have curvature -1. The hyperbolic distance between $a,\ b \in \mathbb{D}$ is

$$d_{\mathbb{D}}(a, b) = 2 \tanh^{-1} \left| \frac{a - b}{1 - \bar{a}b} \right|.$$

Hyperbolic geodesics exist in $\mathbb{D}$ and are arcs of circles orthogonal to the unit circle. The hyperbolic metric on the unit disk is invariant under $\mathrm{Aut}(\mathbb{D})$, the group of conformal automorphisms of $\mathbb{D}$; this means $f^*(\lambda_{\mathbb{D}}(z)|dz|) = \lambda_{\mathbb{D}}(z)|dz|$ for all conformal automorphisms f of $\mathbb{D}$. It is also invariant under all conjugate conformal mappings of $\mathbb{D}$. A region Ω on the Riemann sphere $\mathbb{P}$ is *hyperbolic* if $\mathbb{P} \setminus \Omega$ contains at least three points. If $f : \mathbb{D} \to \Omega$ is any holomorphic covering projection, then the density λ_Ω of the hyperbolic metric $\lambda_\Omega(z)|dz|$ is determined from $\lambda_\Omega(f(z))|f'(z)| = 2/(1-|z|^2)$. In particular, if $f(0) = a$ and $f'(0) > 0$, then $\lambda_\Omega(a) = 2/f'(0)$. It is independent of the choice of covering projection. The hyperbolic metric on a hyperbolic region has curvature -1 and is complete.

It is helpful to have an explicit expression for the hyperbolic metric on several basic regions. The hyperbolic metric on the Euclidean disk $D(a,r) = \{z : |z-a| < r\}$ is

$$\lambda_{D(a,r)}(z)|dz| = \frac{2r|dz|}{r^2 - |z-a|^2}.$$

In particular, $\lambda_{D(a,r)}(a) = 2/r$ and $\lambda_{D(a,r)} \to 0$ as $r \to \infty$. The hyperbolic metric on a half-plane H coincides with the quasihyperbolic metric:

$$\lambda_H(z)|dz| = \frac{|dz|}{\operatorname{dist}(z, \partial H)} = \frac{|dz|}{\delta_H(z)}.$$

If $E(a,r) = \{z \in \mathbb{C} : |z-a| > r\} \cup \{\infty\}$, then

$$\lambda_{E(a,r)}(z)|dz| = \frac{2r|dz|}{|z-a|^2 - r^2},$$

and the hyperbolic metric on the slit plane $\Omega = \mathbb{C} \setminus (-\infty, 0]$ is

$$\lambda_\Omega(z)|dz| = \frac{|dz|}{2|\sqrt{z}|\operatorname{Re}\sqrt{z}}.$$

Lastly, the hyperbolic metric on the punctured unit disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$ is

$$\lambda_{\mathbb{D}^*}(z)|dz| = \frac{|dz|}{|z|\log(1/|z|)}.$$

We present three important properties of the hyperbolic metric that will be used later.

MONOTONICITY: If $\Omega \subseteq \Delta$, then $\lambda_\Delta \le \lambda_\Omega$. Moreover, strict inequality holds unless $\Omega = \Delta$.

INVARIANCE UNDER COVERINGS: If $f : \Omega \to \Delta$ is an unbranched, unlimited (conjugate) holomorphic covering, then $f^*(\lambda_\Delta(z)|dz|) = \lambda_\Omega(z)|dz|$. In particular, the hyperbolic metric is invariant under (conjugate) conformal mappings.

CONTINUOUS DEPENDENCE ON REGION: If $\{\Omega_n\}$ is a sequence of hyperbolic regions converging to the hyperbolic region Ω in the sense of kernel convergence, then $\lambda_{\Omega_n} \to \lambda_\Omega$. (See [4].)

§3 Basic properties of the K-P metric

Several properties of the K-P metric follow easily from the definition.

Theorem 3.1 (Möbius invariance). *Suppose* Ω *is a region in* $\mathbb{P}$ *with at least two boundary points,* T *is a Möbius transformation and* J *is a conjugate Möbius transformation. Then* $T^*(\mu_{T(\Omega)}(w)|dw|) = \mu_\Omega(z)|dz|$ *and* $J^*(\mu_{J(\Omega)}(w)|dw|) = \mu_\Omega(z)|dz|$.

Proof This follows from the fact that a (conjugate) Möbius transformation maps a disk in $\mathbb{P}$ onto a disk together with the invariance of the hyperbolic metric under (conjugate) conformal mappings.

Theorem 3.2 (Monotonicity). *If* $\Omega \subseteq \Delta$, *then* $\mu_\Delta(z)|dz| \leq \mu_\Omega(z)|dz|$.

Proof If D is a disk and $a \in D \subseteq \Omega$, then $a \in D \subseteq \Delta$.

The K-P metric and the quasihyperbolic metric are comparable on any proper subregion of $\mathbb{C}$.

Theorem 3.3. *If* Ω *is a proper subregion of* $\mathbb{C}$, *then*

$$\frac{1}{\delta_\Omega(z)} \leq \mu_\Omega(z) \leq \frac{2}{\delta_\Omega(z)}.$$

Proof Fix $a \in \Omega$. First we establish the upper bound. For $D = D(a, \delta_\Omega(a))$, the disk with center a and radius $\delta_\Omega(a)$, $\lambda_D(a) = 2/\delta_\Omega(a)$. Thus, $\mu_\Omega(a) \leq 2/\delta_\Omega(a)$.

The lower bound is also elementary. Suppose $D = D(c, r)$ is any disk in Ω containing a. Then

$$\lambda_D(a) = \frac{2r}{r^2 - |a-c|^2} > \frac{1}{r - |a-c|} = \frac{1}{\delta_D(a)} \geq \frac{1}{\delta_\Omega(a)}.$$

Similarly, if H is a half-plane in Ω containing a, then

$$\lambda_H(a) = \frac{1}{\delta_H(a)} \geq \frac{1}{\delta_\Omega(a)}.$$

The two preceding inequalities imply $\mu_\Omega(a) \geq 1/\delta_\Omega(a)$.

Next, we derive an explicit formula for the K-P metric in a twice-punctured sphere. Given a, $b \in \mathbb{P}$, let $\mu_{a,b}(z)|dz|$ denote the K-P metric on $\mathbb{P} \setminus \{a, b\}$.

Theorem 3.4. *For distinct points* a, $b \in \mathbb{P}$,

$$\mu_{a,b}(z)|dz| = \frac{|a-b|}{|z-a||z-b|}|dz|;$$

so, $\mu_{a,b}(z)|dz|$ is complete and has curvature zero. In addition, there is a unique extremal disk for each $z \in \mathbb{P} \setminus \{a, b\}$; it is the disk containing z and bounded by the circle through a and b that is orthogonal to the circle through a, b, and z.

Proof We first consider the case $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ and show $\mu_{0,\infty}(z)|dz| = \dfrac{|dz|}{|z|}$. From Theorem 3.3

$$\mu_{0,\infty}(z) \geq \frac{1}{\delta_{0,\infty}(z)} = \frac{1}{|z|}.$$

For the half-plane H_z containing z and bounded by the line through 0 that is orthogonal to the line segment $[0, z]$, $\lambda_{H_z}(z) = 1/\delta_{H_z}(z) = 1/|z|$. Therefore, $\mu_{0,\infty}(z)|dz| = \frac{|dz|}{|z|}$, which is the logarithmic metric on $\mathbb{C}^*$, has curvature 0 and is complete. In fact, the half-plane H_z is the unique extremal disk for the K-P metric at z. This can be seen as follows. If D is a Euclidean disk in $\mathbb{C}^*$ containing z, then

$$\lambda_D(z) > \frac{1}{\delta_D(z)} \geq \frac{1}{\delta_{\mathbb{C}^*}(z)} = \frac{1}{|z|}.$$

If H is any half-plane in $\mathbb{C}^*$ containing z, then

$$\lambda_H(z) = \frac{1}{\delta_H(z)} \geq \frac{1}{\delta_{\mathbb{C}^*}(z)} = \frac{1}{|z|}$$

and equality holds if and only if $H = H_z$.

Next, the Möbius invariance of the K-P metric enables us to determine the K-P metric for $\mathbb{P} \setminus \{a, b\}$. The Möbius transformation $T(z) = (z - a)/(z - b)$ maps $\mathbb{P} \setminus \{a, b\}$ onto $\mathbb{C}^*$, so

$$\mu_{a,b}(z)|dz| = \mu_{0,\infty}(T(z))|T'(z)||dz| = \frac{|a-b|}{|z-a||z-b|}|dz|.$$

Our description (in the theorem) of the extremal disk for $\mathbb{P} \setminus \{a, b\}$ is just the Möbius invariant description of the half-plane H_z. This description is also valid if a or b is ∞.

We point out that when a, $b \in \mathbb{C}$, the disk D with diameter $[a, b]$ is extremal for each point $z \in (a, b) \subset D \subset \mathbb{P} \setminus \{a, b\}$.

The metric $\mu_{a,b}(z)|dz|$ arises naturally from cross-ratios and was defined in this special case by Ferrand [1]. The absolute cross-ratio of four points in $\mathbb{C}$ is

$$|a, z, b, w| = \frac{|a-b||z-w|}{|a-z||b-w|}.$$

Then

$$\lim_{w \to z} \frac{|a, z, b, w|}{|z-w|} = \frac{|a-b|}{|a-z||b-z|} = \mu_{a,b}(z).$$

The invariance of the absolute cross-ratio under Möbius and conjugate Möbius transformations immediately gives the corresponding invariance of $\mu_{a,b}(z)|dz|$.

Now, we prove that the infimum in the definition of the K-P metric can be replaced by a minimum. The uniqueness of extremal disks will be given in Theorem 4.6.

Theorem 3.5 (Existence of an extremal disk). *For any region Ω in $\mathbb{P}$ with at least two boundary points and each point in Ω, an extremal disk exists and must meet $\partial\Omega$. In particular,* $\mu_\Omega(z) = \min\{\lambda_D(z) : z \in D \subseteq \Omega,\ D \text{ is a disk on } \mathbb{P}\}$.

Proof The monotonicity property of the K-P metric together with Theorem 3.4 implies that the K-P metric is nontrivial on any region with at least two boundary points. By performing a Möbius transformation if necessary, we may assume $\Omega \subset \mathbb{C}$. Fix $a \in \Omega$. We show there is a disk D such that $a \in D \subseteq \Omega$ and $\mu_\Omega(a) = \lambda_D(a)$. From the definition of $\mu_\Omega(a)$, there exists a sequence $\{D_n\}$ of disks in $\mathbb{C}$ with $a \in D_n$ such that $\lim_{n\to\infty} \lambda_{D_n}(a) = \mu_\Omega(a)$. Let $g_n : \mathbb{D} \to D_n$ be the Möbius transformation with $g_n(0) = a$ and $g_n'(0) > 0$. Then $\lambda_{D_n}(a) = 2/g_n'(0)$ and $\lim_{n\to\infty} g_n'(0) = 2/\mu_\Omega(a)$. Define $h_n(z) = (g_n(z) - a)/g_n'(0)$. Then $h_n(z)$ is a normalized ($h_n(0) = h_n'(0) - 1 = 0$) Möbius transformation. Since the collection of all normalized holomorphic Möbius transformations in $\mathbb{D}$ is a normal family, there exists a subsequence $\{h_{n_k}\}$ of $\{h_n\}$ which converges to a normalized Möbius transformation h uniformly on every compact subset of $\mathbb{D}$. We write $\{h_n\}$ instead of $\{h_{n_k}\}$, and $\{g_n\}$ instead of $\{g_{n_k}\}$, where $g_n(z) = a + g_n'(0)h_n(z)$ converges to $g(z) = a + (2/\mu_\Omega(a))h(z)$ uniformly on each compact subset of $\mathbb{D}$. Thus, $g(0) = a$, $g'(0) = 2/\mu_\Omega(a)$ and $g(\mathbb{D})$ is the Caratheodory kernel of $\{g_n(\mathbb{D})\}$ with respect to a. Moreover, $D = g(\mathbb{D})$ is a disk on $\mathbb{P}$ containing a, $\lambda_D(a) = 2/g'(0) = \mu_\Omega(a)$ and the Caratheodory kernel $D = g(\mathbb{D})$ of $\{D_n\}$ is contained in Ω since each D_n is contained in Ω. Finally, an extremal disk D meets the boundary of the region: if not, then there would be a larger disk D', properly containing D but still in Ω, and we would obtain the contradiction $\lambda_{D'}(a) < \lambda_D(a) = \mu_\Omega(a)$.

Theorem 3.6. *Suppose Ω is a hyperbolic region in $\mathbb{P}$. If Ω is a disk, then $\lambda_\Omega = \mu_\Omega$. Otherwise, $\lambda_\Omega < \mu_\Omega$ in Ω.*

Proof Domain monotonicity of the hyperbolic metric easily provides the inequality $\lambda_\Omega \le \mu_\Omega$. When Ω is a disk, the definition of μ_Ω yields the oppposite inequality, so $\mu_\Omega = \lambda_\Omega$ when Ω is a disk.

Assume Ω is not a disk. We prove that $\lambda_\Omega < \mu_\Omega$. Suppose there exists $a \in \Omega$

with $\lambda_\Omega(a) = \mu_\Omega(a)$. Let D be an extremal disk for a, so $a \in D \subseteq \Omega$ and $\mu_\Omega(a) = \lambda_D(a)$. Then $D \subseteq \Omega$ and $\lambda_\Omega(a) = \lambda_D(a)$, so $D = \Omega$ by the monotonicity property of the hyperbolic metric.

We gather together some elementary properties of extremal disks.

Theorem 3.7. *Suppose Ω is a region in $\mathbb{P}$ with at least two boundary points.*
(a) *Let D be an extremal disk for $a \in \Omega$. For any Möbius transformation T, $T(D)$ is an extremal disk for $T(a) \in T(\Omega)$. A similar result holds for conjugate Möbius transformations.*
(b) *Let D be an extremal disk for $a \in \Omega$. There is a Möbius transformation T such that $T(D)$ is an extremal disk for $T(a) \in T(\Omega)$ and $T(a)$ is the center of $T(D)$.*
(c) *Suppose $\Omega \subseteq \Delta$ and D is an extremal disk for $\mu_\Delta(z)|dz|$ at $a \in \Delta$. If $D \subseteq \Omega$, then $\mu_\Omega(a) = \lambda_D(a)$, that is, D is also an extremal disk for $\mu_\Omega(z)|dz|$ at $a \in \Omega$. In particular, $\mu_\Omega(a) = \mu_\Delta(a)$ at such points.*

Proof (a) This is a simple consequence of the Möbius invariance of the K-P and hyperbolic metrics.
(b) From Theorem 3.5, an extremal disk D exists for $a \in \Omega$. In general, D need not have center a. If a is the center of D, then there is nothing to do. Otherwise, let $\Gamma = \partial D$ and let a^* be the reflection of a in Γ. Then Γ is an Apollonius circle relative to a and a^*, say $\Gamma = \{z : |(z-a)/(z-a^*)| = k\}$. If $T(z) = (z-a)/(z-a^*)$, then $T(D)$ is an extremal disk for $0 = T(a) \in T(\Omega)$ and $T(D)$ is a disk centered at the origin.
(c) This follows easily from monotonicity of the K-P metric.

Theorem 3.8 (Completeness). *For any region on the sphere with at least two boundary points, the K-P metric is complete.*

Proof If $\mathbb{P}\backslash\Omega$ contains exactly two points, the conclusion follows from Theorem 3.4. Therefore, we need only consider the case of hyperbolic regions and may assume $\Omega \subset \mathbb{C}$. As $\lambda_\Omega \le \mu_\Omega$, any Cauchy sequence in Ω relative to $\mu_\Omega(z)|dz|$ is also a Cauchy sequence relative to $\lambda_\Omega(z)|dz|$. Since $\lambda_\Omega(z)|dz|$ is complete, the sequence converges to some point $a \in \Omega$ relative to the hyperbolic distance. Therefore, the sequence also converges to a relative to the Euclidean topology and so in the distance induced by $\mu_\Omega(z)|dz|$.

Next, we show that the K-P metric depends continuously on the region. We write $\Omega_n \to \Omega$ to mean that the sequence $\{\Omega_n\}$ converges to Ω in the sense of kernel convergence with respect to some base point.

Theorem 3.9 (Continuous dependence on the region). *Suppose* $\{\Omega_n\}_{n=1}^{\infty}$ *is a sequence of hyperbolic regions on* $\mathbb{P}$ *and* $\Omega_n \to \Omega$, *a hyperbolic region. Then for* $z \in \Omega$, $\lim_{n\to\infty} \mu_{\Omega_n}(z) = \mu_\Omega(z)$.

Proof Fix $a \in \Omega$. We first prove $\limsup_{n\to\infty} \mu_{\Omega_n}(a) \leq \mu_\Omega(a)$. Let D be an extremal disk for a, so $a \in D \subseteq \Omega$ and $\mu_\Omega(a) = \lambda_D(a)$. Consider any proper subdisk Δ of D with $a \in \Delta$. Then $\overline{\Delta}$ is a compact subset of D, so there exists N such that $\overline{\Delta} \subset \Omega_n$ for all $n \geq N$. Hence, for $n \geq N$, $\mu_{\Omega_n}(a) \leq \lambda_\Delta(a)$, and so $\limsup_{n\to\infty} \mu_{\Omega_n}(a) \leq \lambda_\Delta(a)$. By letting the disk Δ approach D, we obtain $\limsup_{n\to\infty} \mu_{\Omega_n}(a) \leq \lambda_D(a) = \mu_\Omega(a)$.

Next, we prove $\mu_\Omega(a) \leq m = \liminf_{n\to\infty} \mu_{\Omega_n}(a)$. We assume $\lim_{n\to\infty} \mu_{\Omega_n}(a) = m$; if not, we pass to an appropriate subsequence. Let D_n be an extremal disk for $a \in \Omega_n$, so $D_n \subseteq \Omega_n$ and $\mu_{\Omega_n}(a) = \lambda_{D_n}(a)$. Consider the Möbius transformation $T_n : \mathbb{D} \to D_n$ with $T_n(0) = a$ and $T_n'(0) > 0$. Then $\lambda_{D_n}(a)T_n'(0) = \lambda_{\mathbb{D}}(0) = 2$, or

$$T_n'(0) = \frac{2}{\lambda_{D_n}(a)} = \frac{2}{\mu_{\Omega_n}(a)}.$$

From $\Omega_n \to \Omega$, we obtain $\lambda_{\Omega_n}(a) \to \lambda_\Omega(a)$ [4]. Because $\lambda_{\Omega_n}(a) \leq \mu_{\Omega_n}(a)$, we get

$$0 < \lambda_\Omega(a) = \lim_{n\to\infty} \lambda_{\Omega_n}(a) \leq \lim_{n\to\infty} \mu_{\Omega_n}(a) = m.$$

Also, from what we have proved in the other direction, we have

$$m = \lim_{n\to\infty} \mu_{\Omega_n}(a) = \limsup_{n\to\infty} \mu_{\Omega_n}(a) \leq \mu_\Omega(a).$$

Hence, $0 < \lambda_\Omega(a) \leq m \leq \mu_\Omega(a) < \infty$. Since $\lim_{n\to\infty} \mu_{\Omega_n}(a) = m \neq 0,\ \infty$, there exists M so that $1/M \leq T_n'(0) \leq M$ for all n. This implies that the sequence $\{T_n\}_{n=1}^{\infty}$ is a normal family, so we may assume $T_n \to T$ locally uniformly on $\mathbb{D}$. The limit function T is not a constant since $\frac{1}{M} \leq T'(0) \leq M$. Therefore, T is a Möbius transformation with $T(0) = a$ and $T'(0) > 0$. In fact, $T'(0) = 2/m$ since

$$\frac{2}{\mu_{\Omega_n}(a)} = T_n'(0) \to T'(0).$$

As $T_n(\mathbb{D}) = D_n \subseteq \Omega_n$, $\Omega_n \to \Omega$ and $T_n \to T$, it follows that $D = T(\mathbb{D}) \subseteq \Omega$. Then

$$\mu_\Omega(a) \leq \lambda_D(a) = \frac{2}{T'(0)} = m.$$

Hence, $\mu_\Omega(a) \leq m = \liminf_{n\to\infty} \mu_{\Omega_n}(a)$.

§4 Characterization of extremal disks

It is possible to obtain precise information about extremal disks. We begin by introducing the notion of the closed hyperbolically convex hull of a compact subset on the boundary of a disk. Suppose D is a disk in $\mathbb{P}$ with boundary circle Γ and K is a non-empty compact subset of Γ. Recall that each hyperbolic geodesic of D divides D into two hyperbolic half-planes. By adding the hyperbolic geodesic to either of these we obtain a closed hyperbolic half-plane in D; here "closed" is relative to the induced spherical topology on D. Letting H denote such closed hyperbolic half-planes in D, we define

$$\widehat{K} = \bigcap \{H : H \subset D \text{ and the spherical closure of } H \text{ contains } K\};$$

this is a closed hyperbolically convex subset of D which we call the *hyperbolically convex hull of* K. Clearly, $\widehat{K}$ is Möbius invariant. If $K = \{a\}$, then $\widehat{K} = \emptyset$. The opposite extreme is when $K = \Gamma$; in this case we set $\widehat{K} = D$. For $K = \{a, b\}$, $\widehat{K}$ is the hyperbolic geodesic in D ending at a and b. If $K = \{a, b, c\}$, then $\widehat{K}$ is the ideal hyperbolic triangle with vertices a, b, c. If K contains n points, $n \geq 3$, then $\widehat{K}$ is the closed ideal hyperbolic n-gon with vertices at the points of K.

We already know extremal disks must meet the boundary. Now we refine this result and prove that every maximal disk (in the sense of set inclusion) in Ω is an extremal disk. Evidently, a maximal disk in Ω meets the boundary at least twice.

Theorem 4.1 (Characterization of extremal disks). *Let Ω be a region on $\mathbb{P}$ with at least two boundary points.*
(a) *If D is an extremal disk for $a \in \Omega$ and $K = \partial D \cap \partial \Omega$, then $a \in \widehat{K}$. In particular, K must contain at least two points.*
(b) *If D is a disk in Ω such that $K = \partial D \cap \partial \Omega$ contains at least two points, then D is an extremal disk for all $a \in \widehat{K}$.*

Proof We prove part (a) now; the proof of part (b) is delayed until we establish Theorem 4.2 and Lemma 4.3. Suppose D is an extremal disk for $a \in \Omega$. Since the boundary of an extremal disk must meet $\partial\Omega$, $K \neq \emptyset$. We show D is not an extremal disk if $a \notin \widehat{K}$. By Theorem 3.7(b), we may take the extremal disk to be $\mathbb{D}$ and $a = 0$ the center of $\mathbb{D}$. Since $0 \notin \widehat{K}$, there is a closed hyperbolic half-plane $H \subset \mathbb{D}$ with $0 \in \partial H$ and $H \cap \widehat{K} = \emptyset$. By rotating the disk if necessary, we may assume that $H = \mathbb{D} \cap \{z : \operatorname{Re} z \leq 0\}$, so $K \subset \partial\mathbb{D} \cap \{z : \operatorname{Re} z > 0\}$. Because K is compact, there exists θ, $0 < \theta < \pi/2$, so that $K \subset \{e^{i\varphi} : -\theta < \varphi < \theta\}$. The complementary closed arc $\Gamma_\theta = \partial\mathbb{D} \setminus \{e^{i\varphi} : -\theta < \varphi < \theta\}$ is a compact subset of Ω since $K = \partial\mathbb{D} \cap \partial\Omega$. Hence, there exists $\delta > 0$ so that $\{z : \mathrm{dist}\,(z, \Gamma_\theta) < \delta\} \subset \Omega$. This implies that for sufficiently small $\varepsilon > 0$, the disk Δ with center $-\varepsilon$ and radius $r_\varepsilon = |e^{i\theta} - (-\varepsilon)| = |e^{i\theta} + \varepsilon| > 1$ is contained in Ω and $0 \in \Delta$. From $r_\varepsilon^2 = 1 + 2\varepsilon\cos\theta + \varepsilon^2$, we have $r_\varepsilon^2 - \varepsilon^2 = 1 + 2\varepsilon\cos\theta$. Hence,

$$\lambda_\Delta(0) = \frac{2r_\varepsilon}{r_\varepsilon^2 - \varepsilon^2} = \frac{2\sqrt{1 + 2\varepsilon\cos\theta + \varepsilon^2}}{1 + 2\varepsilon\cos\theta},$$

while $\lambda_{\mathbb{D}}(0) = 2$. Since $0 < \theta < \pi/2$, the inequality $\varepsilon - 4\varepsilon\cos^2\theta < 2\cos\theta$ holds for sufficiently small ε. This is equivalent to $\sqrt{1 + 2\varepsilon\cos\theta + \varepsilon^2} < 1 + 2\varepsilon\cos\theta$. Thus, $\lambda_\Delta(0) < \lambda_{\mathbb{D}}(0)$ holds for sufficiently small ε, so $\mathbb{D}$ is not an extremal disk for $a = 0 \in \Omega$, a contradiction.

Remark. If $\partial D \cap \partial\Omega$ contains exactly two points, then a lies on the hyperbolic geodesic of D ending at these two points of intersection. As we have seen, this is the case for a twice-punctured sphere.

Now, we establish an explicit formula for the K-P metric $\mu_{a,b,c}(z)|dz|$ for the thrice-punctured sphere $\mathbb{P} \setminus \{a, b, c\}$. First, we consider the special case $\mathbb{C} \setminus \{-1, 1\}$.

Theorem 4.2. *For $z \in \mathbb{C} \setminus \{-1, 1\}$,*

$$\mu_{1,-1,\infty}(z) = \begin{cases} 2/|z^2 - 1| & \text{if } z \in \mathbb{D}, \\ 1/|z - 1| & \text{if } \mathrm{Re}\, z > 1, \\ 1/|z + 1| & \text{if } \mathrm{Re}\, z < -1, \\ 1/\mathrm{Im}\, z & \text{if } z \in \Delta, \\ -1/\mathrm{Im}\, z & \text{if } z \in \Delta^*, \end{cases}$$

where $\Delta = \{z : |\mathrm{Re}\, z| \le 1,\ \mathrm{Im}\, z > 0 \text{ and } |z| \ge 1\}$ and Δ^* is the reflection of Δ over the real axis. Moreover, the extremal disks for $\mu_{1,-1,\infty}(z)|dz|$ are unique.

Proof Let $\Omega = \mathbb{C} \setminus \{-1, 1\}$. If $z \in \mathbb{D}$, we use the fact that $\Omega \subset \mathbb{P} \setminus \{-1, 1\}$. When we regard z as a point in $\mathbb{P} \setminus \{-1, 1\}$, the extremal disk for $\mu_{-1,1}(z)$ does not contain ∞ by Theorem 3.4, so it lies in Ω. Thus, Theorem 3.4 and Theorem 3.7(c) imply

$$\mu_\Omega(z) = \mu_{-1,1}(z) = \frac{2}{|z - 1||z + 1|}.$$

If $z \in \Omega$ and $\mathrm{Re}\, z > 1$, we use the fact $\Omega \subset \mathbb{C} \setminus \{1\}$. In $\mathbb{C} \setminus \{1\}$, the extremal half-plane for those z with $\mathrm{Re}\, z \ge 1$ does not contain -1. Hence,

$$\mu_\Omega(z) = \mu_{1,\infty}(z) = 1/|z - 1|.$$

Since $z \mapsto -z$ is an isometry of $\mu_\Omega(z)|dz|$,

$$\mu_\Omega(z) = \mu_{-1,\infty}(z) = 1/|z+1|$$

for $z \in \Omega$ with $\operatorname{Re} z < 1$. Now, we consider $z \in \Delta$. By Theorem 4.1(a) for an extremal disk D, $K = \partial D \cap \partial\Omega$ contains at least two points and $z \in \widehat{K}$. For $z \in \Delta$, no Euclidean disk D with -1 and 1 on its boundary can be extremal for z because z does not lie on the circular arc through -1 and 1 that is orthogonal to ∂D. Likewise, any half-plane H containing z with $1 \in \partial H$ and $-1 \notin \partial H$ cannot be extremal either since z does not lie along the ray from 1 that is orthogonal to ∂H. The same argument applies to any half-plane H containing z with $1 \notin \partial H$ and $-1 \in \partial H$. The only remaining possibility for the extremal disk is the upper half-plane $\mathbb{H}$, which has all three boundary points -1, 1, ∞ of $\mathbb{C} \setminus \{-1, 1\}$ on its boundary. Thus,

$$\mu_\Omega(z) = \lambda_{\mathbb{H}}(z) = 1/\operatorname{Im} z.$$

The case $z \in \Delta^*$ follows from the preceding case since $z \mapsto \bar{z}$ is an isometry of $\mu_\Omega(z)|dz|$.

It remains to demonstrate that extremal disks are unique for $\Omega = \mathbb{C} \setminus \{1, -1\}$. In the above we determined extremal disks for μ_Ω; now we use Theorem 4.1(a) to show these are the only extremal disks. By this result, an extremal disk D for a point z must meet $\partial\Omega$ at least twice and $z \in \widehat{K}$, where $K = \partial D \cap \partial\Omega$. The only possible extremal disks are disks bounded by circles through 1 and -1, through 1 and ∞, through -1 and ∞ or the circle through 1, -1 and ∞. If a Euclidean disk D has 1 and -1 as boundary points, then it is extremal only for points in D lying on the circular arc δ in D ending at ± 1 that is orthogonal to ∂D. These arcs are distinct for different disks and foliate $\mathbb{D}$. Next, consider half-planes H with 1 on the boundary and $-1 \notin H$. This is extremal only for points on the ray emanating from 1 and orthogonal to the edge of H. These rays foliate the half-plane $\{z : \operatorname{Re} z > 1\}$. Similarly, the half-planes that have -1 on the boundary and do not contain 1 give rise to a foliation of the half-plane $\{z : \operatorname{Re} z < -1\}$. Next, consider the upper half-plane which is bounded by the circle containing 1, -1 and ∞. Then $\mathbb{H}$ is extremal for all points in the closed ideal hyperbolic triangle with vertices 1, -1 and ∞. Finally, the situation is similar for $\mathbb{H}^*$.

Notice that $\mu_{1,-1,\infty}(z)|dz|$ has curvature 0 in the first three cases in Theorem 4.2 and curvature -1 in the interior of Δ and Δ^*. Also, the proof contains a complete description of the extremal disks.

Now, we restate the result of the preceding theorem in a Möbius invariant formulation. Suppose a, b, c are distinct points in $\mathbb{P}$ and $\Omega = \mathbb{P}\setminus\{a,b,c\}$. Denote by Γ the unique circle through a, b, c. Let $\gamma_{a,b}$ be the unique circle through a, b that is orthogonal to Γ; the circles $\gamma_{b,c}$ and $\gamma_{c,a}$ are defined similarly. The three circles $\gamma_{a,b}$, $\gamma_{b,c}$ and $\gamma_{c,a}$ are invariant under reflection in Γ, and each one is tangent to the other two. Let $D_{a,b}$ be the component of $\mathbb{P}\setminus\gamma_{a,b}$ that contains the arc between a and b of Γ; $D_{b,c}$ and $D_{c,a}$ are defined similarly. In the complement of Γ, the two components of $\mathbb{P}\setminus(D_{a,b}\cup D_{b,c}\cup D_{c,a})$ are closed ideal hyperbolic triangles Δ and Δ^*, which are interchanged by reflection in Γ. In $D_{a,b}$,

$$\mu_{a,b,c}(z) = \mu_{a,b}(z) = \frac{|a-b|}{|z-a||z-b|}$$

with analogous results in $D_{b,c}$ and $D_{c,a}$. If D is the component of $\mathbb{P}\setminus\Gamma$ containing Δ, then D is the extremal disk for each $z\in\Delta$ and $\mu_{a,b,c}(z) = \lambda_D(z)$.

Here is the last piece of information we require to complete our proof of Theorem 4.1.

Lemma 4.3. *Suppose D is a disk on $\mathbb{P}$ and K is a compact subset of ∂D containing at least two points. If $z\in\widehat{K}$, then either*
(i) *there exist a, $b\in K$ with z on the hyperbolic geodesic in D joining a and b, or*
(ii) *there exist a, b, $c\in K$ with z inside the ideal hyperbolic triangle with vertices a, b, c.*

Proof Because of the Möbius invariance of the situation, there is no loss of generality in assuming $D=\mathbb{D}$, $z=0$ and $1\in K$. If K contains a pair a and $b=-a$ of diametrically opposite points, then we are in case (i). If K does not contain a pair of diametrically opposite points, we show that (ii) holds. Choose $b\in K$ so $|b-1| = \max\{|z-1| : z\in K\}$. Since $1\in K$ and K does not contain a pair of diametrically opposite points, $|b-1|<2$. Also, $|b-1|\geq\sqrt{2}$ since $0\in\widehat{K}$. Because the result is invariant under reflection in the real axis, we may assume $\operatorname{Im} b>0$. Then $b=e^{i\theta}$, where $\pi/2\leq\theta<\pi$. By hypothesis, $-b\notin K$. Since $0\in\widehat{K}$, there must be a point c of K on the open arc of the unit circle in the lower half-plane between $-b$ and -1. Then 0 lies in the interior of the ideal hyperbolic triangle with vertices 1, b and c.

We now complete the proof of Theorem 4.1.

Proof (of Theorem 4.1(b)) Let D be a disk as in the statement of Theorem 4.1(b) and $z\in\widehat{K}$. By Lemma 4.3, we need only consider two cases.

(i) First, assume there exist a, $b \in K$ such that z is on the hyperbolic geodesic δ in D joining a and b. In this case, $\Omega \subseteq \mathbb{P} \setminus \{a, b\}$. By Theorem 3.4, D is an extremal disk for $\mu_{a,b}(z)$. Theorem 3.7(c) implies D is an extremal disk for $\mu_\Omega(z)$ and $\mu_\Omega(z) = \lambda_D(z) = \mu_{a,b}(z)$.

(ii) Next, assume there exist a, b $c \in K$ so z lies in the ideal hyperbolic triangle with vertices a, b and c. Now $\Omega \subseteq \mathbb{P} \setminus \{a, b, c\}$ and D is an extremal disk for $\mu_{a,b,c}(z)$. Theorem 3.7(c) implies D is an extremal disk for $\mu_\Omega(z)$ and $\mu_\Omega(z) = \lambda_D(z) = \mu_{a,b,c}(z)$.

Example 4.4 For the punctured unit disk $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$,

$$\mu_{\mathbb{D}^*}(z) = \frac{1}{|z|(1 - |z|)}.$$

The K-P metric for $\mathbb{D}^*$ is invariant under rotations about the origin, so it suffices to consider the case $z = x \in (0, 1)$. The disk $D = D(\frac{1}{2}, \frac{1}{2})$ lies in the punctured unit disk and meets the boundary at the origin and 1, so it is extremal for all $z = x \in (0, 1)$. For $0 < x < 1$,

$$\lambda_D(x) = \frac{1}{\frac{1}{2}^2 - |x - \frac{1}{2}|^2} = \frac{1}{x - x^2}.$$

Example 4.5 For $\Omega = \mathbb{C} \setminus (-\infty, 0]$,

$$\mu_\Omega(re^{i\theta}) = \begin{cases} \frac{1}{r|\sin\theta|} & \text{if } \pi/2 \le |\theta| < \pi \\ \frac{1}{r} & \text{if } |\theta| < \pi/2. \end{cases}$$

The upper half-plane $\mathbb{H}$ lies in Ω and meets the boundary of Ω along the negative real axis, including ∞. The closed hyperbolically convex hull of this boundary intersection in $\mathbb{H}$ is $\{re^{i\theta} : \pi/2 \le \theta < \pi\}$. Hence, for $z = re^{i\theta}$, $\pi/2 \le \theta < \pi$,

$$\mu(z) = \lambda_{\mathbb{H}}(z) = \frac{1}{\operatorname{Im} z} = \frac{1}{r\sin\theta}.$$

A similar result holds with the lower half-plane replacing the upper half-plane. The remaining case is $z = re^{i\theta}$, where $|\theta| < \pi/2$. The half-plane H that contains z and bounded by the line through the origin orthogonal to the segment $[0, z]$ is extremal, so $\mu_\Omega(re^{i\theta}) = \lambda_H(z) = 1/r$.

Theorem 4.6 (Uniqueness of extremal disks). *Suppose Ω is a region in $\mathbb{P}$ such that $\mathbb{P} \setminus \Omega$ contains at least two points. Then extremal disks for the K-P metric are unique.*

Proof From Theorems 3.4 and 4.2 we know that extremal disks are unique for a twice-punctured or thrice-puntured sphere. Now we turn to the general case.

Suppose D_z is an extremal disk for $z \in \Omega$. Put $K = \partial D_z \cap \partial\Omega$. By Theorem 4.1(a), $z \in \widehat{K}$ and K contains at least two points. Next, according to Lemma 4.3, either: (i) there exist a, $b \in K$ such that z lies on the hyperbolic geodesic in D_z joining a and b, or, (ii) there exist a, b, $c \in K$ such that z lies inside the ideal hyperbolic triangle in D_z with vertices a, b and c. A close examination of the descriptions of extremal disks given in Theorems 3.4 and 4.2 reveals that in both cases D_z is also the extremal disk for z, in $\mathbb{P} \setminus \{a, b\}$ in case (i) (so $\mu_\Omega(z) = \mu_{a,b}(z)$), or in $\mathbb{P} \setminus \{a, b, c\}$ in case (ii) (so $\mu_\Omega(z) = \mu_{a,b,c}(z)$).

Now suppose D is any extremal disk for $\mu_\Omega(z)$, so $z \in D \subseteq \Omega$ and $\lambda_D(z) = \mu_\Omega(z)$. In case (i), $\mu_\Omega(z) = \mu_{a,b}(z)$. Since D also lies in $\mathbb{P} \setminus \{a, b\}$ and extremal disks for $\mu_{a,b}$ are unique by Theorem 3.4, we conclude $D = D_z$. Similarly, the extremal disk D_z is unique in case (ii) using the fact that extremal disks for $\mu_{a,b,c}$ are unique.

We summarize our results about extremal disks. For each point $z \in \Omega$ there is a unique extremal disk $D_z \subseteq \Omega$ containing z. Moreover, the compact subset $K_z = \partial D_z \cap \partial\Omega$ of the boundary contains at least two points. Let $\widehat{K}_z$ denote the closed hyperbolically convex hull of K_z in D_z. For each $w \in \widehat{K}_z$, $D_w = D_z$ and $\mu_\Omega(w) = \lambda_{D_z}(w)$. For any z, $w \in \Omega$, the sets $\widehat{K}_z$, $\widehat{K}_w$ are either disjoint or identical. Thus, the sets $\{\widehat{K}_z : z \in \Omega\}$ form a partition of Ω. More precisely, these sets form a stratification of Ω, see [5].

§5 Quasi-invariance of the K-P metric

Although the K-P metric is not conformally invariant, it does not change substantially under a conformal mapping. In order to establish this, we prove a result about the family S of normalized univalent functions $f(z) = z + a_2 z^2 + \ldots$ in $\mathbb{D}$.

Theorem 5.1. *If $f \in S$, then there exists a disk or a half-plane D such that $0 \in D \subset f(\mathbb{D})$ and $\lambda_D(0) = 4$.*

Proof Since S and $\lambda_D(0)$ are both invariant under rotations, we only need to prove the result for $f(z) = z + a_2 z^2 + \ldots$ with $0 \le a_2 \le 2$. For any $w \notin f(\mathbb{D})$, define

$$g(z) = \frac{wf(z)}{w - f(z)} = z + \left(a_2 + \frac{1}{w}\right) z^2 + \ldots .$$

Then $g \in S$, so

$$\left|a_2 + \frac{1}{w}\right| \le 2.$$

If $a_2 = 2$, then $f(z) = z/(1-z)^2$, the Koebe function, and $f(\mathbb{D}) = \mathbb{C} \setminus (-\infty, -1/4]$. Hence, the half-plane $D = \{\zeta : \operatorname{Re} \zeta > -1/4\}$ is contained in $f(\mathbb{D})$ and $\lambda_D(0) = 4$.

Now, assume $0 \le a_2 < 2$. From the above inequality, $\frac{1}{w}$ lies in the closed disk centered at $-a_2$ of radius 2; therefore w lies outside the open disk D with center $a_2/(4-a_2^2)$ and radius $2/(4-a_2^2)$. Thus, $0 \in D \subset f(\mathbb{D})$ and we have $\lambda_D(0) = 4$.

The classical Koebe one-quarter theorem gives $D = D(0, 1/4) \subset \mathbb{D}$ and $\lambda_D(0) = 8$. This is weaker than Theorem 5.1.

Theorem 5.2. *Suppose $f : \Omega \to \Delta$ is an unbranched, unlimited (conjugate) holomorphic covering. Then for all $z \in \Omega$,*

$$\frac{1}{2}\mu_\Omega(z) \le \mu_\Delta(f(z))|f'(z)|. \tag{5.3}$$

Moreover, if $f : \Omega \to \Delta$ is a (conjugate) conformal mapping, then

$$\frac{1}{2}\mu_\Omega(z) \le \mu_\Delta(f(z))|f'(z)| \le 2\mu_\Omega(z).$$

These bounds are best possible.

Proof First, we establish inequality (5.3) when $f : \Omega \to \Delta$ is an unbranched, unlimited holomorphic covering. The proof for a conjugate holomorphic covering is similar. Fix $a \in \Omega$ and set $b = f(a)$. From Theorem 3.5, there exists an extremal disk D_0 for $b \in \Delta$. Thus, $D_0 \subseteq \Delta$ and $\mu_\Delta(f(a)) = \mu_\Delta(b) = \lambda_{D_0}(b)$. Let $g = f^{-1}|D_0$ be the branch of the inverse function with $g(b) = a$. Then g is univalent in D_0. If $T : \mathbb{D} \to D_0$ is the Möbius transformation with $T(0) = b$ and $T'(0) > 0$, then $\lambda_{D_0}(b) = 2/T'(0)$. Thus, we need to show $\mu_\Omega(a) \le 4|f'(a)|/T'(0)$ or $\mu_\Omega(a) \le 4/(|g'(b)|T'(0))$ since $g'(b)f'(a) = 1$. Define $h(z) = (g(T(z)) - g(b))/(g'(b)T'(0))$. Then h is univalent in $\mathbb{D}$ with $h(0) = h'(0) - 1 = 0$ and Theorem 5.1 implies that there is a disk $D \subseteq h(\mathbb{D})$ with $\lambda_D(0) = 4$. Set $S(w) = (w-a)/(g'(b)T'(0))$. Then $S^{-1}(D)$ is a disk in Ω and $\lambda_D(S(w))|S'(w)| = \lambda_{S^{-1}(D)}(w)$. In particular, $\lambda_D(0)|S'(a)| = \lambda_{S^{-1}(D)}(a)$, so $4/(|g'(b)|T'(0)) = \lambda_{S^{-1}(D)}(a)$. Hence,

$$\mu_\Omega(a) \le \lambda_{S^{-1}(D)}(a) = \frac{4}{|g'(b)|T'(0)}.$$

For the case when $f : \Omega \to \Delta$ is a conformal mapping, we need only establish the upper bound. This follows by applying the lower bound (5.3) to the inverse function $g = f^{-1} : \Delta \to \Omega$. That the constants are best possible for conformal mappings follows by considering the conformal mapping $f(z) = i\sqrt{z}$ of $\Omega = \mathbb{C} \setminus (-\infty, 0]$ onto

$\mathbb{H}$. Since $\mu_{\mathbb{H}} = \lambda_{\mathbb{H}}$, $\mu_{\mathbb{H}}(f(x))|f'(x)| = 1/(2x)$ for $x > 0$. For $x > 0$, $\mu_\Omega(x) = 1/x$ by Example 4.5. Thus, $\frac{1}{2}\mu_\Omega(x) = \mu_{\mathbb{H}}(f(x))|f'(x)|$ for $x > 0$.

§6 Comparison with the hyperbolic and quasihyperbolic metrics

We know $\lambda_\Omega \leq \mu_\Omega$ for any hyperbolic region. We investigate when these two metrics are equivalent; it is convenient to introduce the domain constant

$$m(\Omega) = \inf\left\{\frac{\lambda_\Omega(z)}{\mu_\Omega(z)} : z \in \Omega\right\}.$$

Theorem 6.1. *Suppose Ω is a proper subregion of $\mathbb{C}$. Then*

$$\frac{1}{2}c(\Omega) \leq m(\Omega) \leq c(\Omega) = \inf\{\delta_\Omega(z)\lambda_\Omega(z) : z \in \Omega\}.$$

In particular, Ω is uniformly perfect if and only if $m(\Omega) > 0$.

Proof This is an immediate consequence of Theorem 3.3.

Example 6.2 For $\mathbb{D}^* = \mathbb{D} \setminus \{0\}$, $\lambda_{\mathbb{D}^*}(z) = 1/|z|\log(1/|z|)$ and $\mu_{\mathbb{D}^*}(z) = 1/|z|(1-|z|)$ by Example 4.4, so

$$\frac{\lambda_{\mathbb{D}^*}(z)}{\mu_{\mathbb{D}^*}(z)} = \frac{1-|z|}{\log(1/|z|)}.$$

This quantity tends to 0 as $|z| \to 0$ and approaches 1 when $|z| \to 1$. Thus, $m(\mathbb{D}^*) = 0$ which is consistent with the fact that $\mathbb{D}^*$ is not uniformly perfect.

Precise bounds on the quotient $\lambda_\Omega/\mu_\Omega$ are available for simply connected regions.

Theorem 6.3. *If Ω is a simply connected hyperbolic region, then*

$$\frac{1}{2} \leq \frac{\lambda_\Omega(z)}{\mu_\Omega(z)} \leq 1.$$

These bounds are best possible.

Proof The upper bound has been verified, and cannot be improved since $\lambda_D = \mu_D$ for disks. In order to establish the lower bound, let $f : \Omega \to \mathbb{D}$ be a conformal mapping. Since the hyperbolic metric is conformally invariant, $\lambda_{\mathbb{D}}(f(z))|f'(z)| =$

$\lambda_\Omega(z)$. By Theorem 5.2 $\mu_{\mathbb{D}}(f(z))|f'(z)| \geq \frac{1}{2}\mu_\Omega(z)$. Since $\mu_{\mathbb{D}} = \lambda_{\mathbb{D}}$, we deduce $\lambda_\Omega(z) \geq \frac{1}{2}\mu_\Omega(z)$. The following example demonstrates the lower bound cannot be improved. For $\Omega = \mathbb{C} \setminus (-\infty, 0]$, we show $\lambda_\Omega(w)/\mu_\Omega(w) \geq 1/2$ with equality if and only if $0 < w < +\infty$. From $\lambda_\Omega(re^{i\theta}) = 1/(2r\cos(\theta/2))$ and Example 4.5

$$\frac{\lambda_\Omega(re^{i\theta})}{\mu_\Omega(re^{i\theta})} = \frac{|\sin(\theta)|}{2\cos(\theta/2)} = |\sin(\theta/2)| \geq \sin(\pi/4) = \sqrt{2}/2$$

for $\pi/2 \leq |\theta| < \pi$. If $|\theta| < \pi/2$, then $\mu_\Omega(re^{i\theta}) = 1/r$ and

$$\frac{\lambda_\Omega(re^{i\theta})}{\mu_\Omega(re^{i\theta})} = \frac{1}{2\cos(\theta/2)} \geq 1/2$$

with equality only on the positive real axis.

If Ω and Δ are conformally equivalent regions, then [6]

$$\frac{1}{\sqrt{1 + 3\coth^2(\pi/3)}} < \frac{c(\Omega)}{c(\Delta)} < \sqrt{1 + 3\coth^2(\pi/3)} = 2.4335\ldots.$$

Harmelin and Minda [3] conjectured: $\frac{1}{2} \leq \frac{c(\Omega)}{c(\Delta)} \leq 2$ when Ω and Δ are conformally equivalent. This conjecture remains an open problem. The analogous result for $m(\Omega)$ holds.

Theorem 6.4. *If $f : \Omega \to \Delta$ is an unbranched, unlimited covering, then $m(\Delta) \leq 2m(\Omega)$. If Ω and Δ are conformally equivalent, then*

$$\frac{1}{2}m(\Omega) \leq m(\Delta) \leq 2m(\Omega).$$

The constants $1/2$ and 2 are best possible.

Proof Suppose $f : \Omega \to \Delta$ is an unbranched, unlimited covering. From Theorem 5.2, $\frac{1}{2}\mu_\Omega(z) \leq \mu_\Delta(f(z))|f'(z)|$. Since $\lambda_\Delta(f(z))|f'(z)| = \lambda_\Omega(z)$, we obtain

$$2\frac{\lambda_\Omega(z)}{\mu_\Omega(z)} \geq \frac{\lambda_\Delta(f(z))|f'(z)|}{\mu_\Delta(f(z))|f'(z)|}.$$

This gives $m(\Delta) \leq 2m(\Omega)$.

The lower bound for conformally equivalent regions follows from the result for coverings by considering the inverse function. The examples $m(\mathbb{D}) = 1$ and $m(\mathbb{C}\setminus(-\infty, 0]) = 1/2$ show that the constants in the theorem are best possible.

§7 Other definitions of the K-P metric

In this section, we present two other expressions for the K-P metric based on extremal problems involving Möbius transformations.

We begin by recalling the definition of the Kobayashi metric; of course in a hyperbolic plane region the Kobayashi metric reduces to the hyperbolic metric, and this provides a description of the latter in terms of an extremal problem. For a hyperbolic region $\Omega \subset \mathbb{C}$ and $a \in \Omega$, consider the family $\mathcal{F}(a, \Omega)$ of all holomorphic functions $f : \mathbb{D} \to \Omega$ normalized by $f(0) = a$ and $f'(0) > 0$. The normalized ($\varphi(0) = a$ and $\varphi'(0) > 0$) holomorphic covering projection $\varphi : \mathbb{D} \to \Omega$ belongs to $\mathcal{F}(a, \Omega)$ and $\lambda_\Omega(a) = 2/\varphi'(0)$. Every function $f \in \mathcal{F}(a, \Omega)$ is subordinate to φ, so there is a holomorphic self-map h of the unit disk that fixes the origin with $h'(0) > 0$ such that $f = \varphi \circ h$. Schwarz' Lemma gives $|h'(0)| \leq 1$ with equality if and only if h is the identity. From $f'(0) = \varphi'(0)h'(0)$ we conclude $f'(0) \leq \varphi'(0)$ for all $f \in \mathcal{F}(a, \Omega)$ with strict inequality unless $f = \varphi$. Hence

$$\lambda_\Omega(a) = \min\left\{\frac{2}{f'(0)} : f \in \mathcal{F}(a, \Omega)\right\},$$

and the extremal function φ is unique.

A number of other metrics are defined by the same type of extremal problem but with a restricted class of admissible functions. For instance, the Hahn metric is defined by replacing $\mathcal{F}(a, \Omega)$ with the subfamily of injective holomorphic mappings. The K-P metric comes from restricting attention to the smaller class of Möbius transformations. For a region Ω in $\mathbb{P}$ such that $\mathbb{P}\backslash\Omega$ contains at least two points and $a \in \Omega$, let $\mathcal{M}(a, \Omega)$ be the family of Möbius transformations satisfying $T(\mathbb{D}) \subseteq \Omega$, $T(0) = a$ and $T'(0) > 0$.

Theorem 7.1. *If Ω is a region in $\mathbb{P}$ with at least two boundary points and $a \in \Omega$, then*

$$\mu_\Omega(a) = \min\left\{\frac{2}{T'(0)} : T \in \mathcal{M}(a, \Omega)\right\}.$$

Moreover, there exists a unique $S \in \mathcal{M}(a, \Omega)$ such that $T'(0) \leq S'(0)$ for all $T \in \mathcal{M}(a, \Omega)$.

Proof Suppose D is the extremal disk for $a \in \Omega$, so $a \in D \subseteq \Omega$ and $\mu_\Omega(a) = \lambda_D(a)$. Let S be the unique Möbius transformation with $S(0) = a$, $S'(0) > 0$ and $S(\mathbb{D}) = D$. Then $\mu_\Omega(a) = \lambda_D(a) = 2/S'(0)$. For any $T \in \mathcal{M}(a, \Omega)$, $T(\mathbb{D})$ is a disk in Ω containing $T(0) = a$, so $\mu_\Omega(a) \le \lambda_{T(\mathbb{D})}(a) = 2/T'(0)$. Hence, $T'(0) \le S'(0)$. Next, S is unique. Suppose $T'(0) = S'(0)$. Then

$$\mu_\Omega(a) = \lambda_{S(\mathbb{D})}(a) = \frac{2}{S'(0)} = \frac{2}{T'(0)} = \lambda_{T(\mathbb{D})}(a),$$

so $T(\mathbb{D})$ is also an extremal disk for $a \in \Omega$. Theorem 4.6 implies that $S(\mathbb{D}) = T(\mathbb{D})$. Since $T^{-1} \circ S$ maps $\mathbb{D}$ onto $\mathbb{D}$ with $T^{-1} \circ S(0) = 0$ and $\left(T^{-1} \circ S\right)'(0) = 1$, it follows that $T = S$.

There is a dual version of the above. For $a \in \mathbb{C}$, let $\mathcal{N}(a)$ denote the family of Möbius transformations normalized by $T(a) = 0$, $T'(a) = 1$. Suppose Ω is a region on $\mathbb{P}$ such that $\mathbb{P}\backslash\Omega$ has at least two points and $a \in \Omega$. We assume $a \ne \infty$ in order to avoid the use of a local coordinate. For $T \in \mathcal{N}(a)$, let $r(T) = \operatorname{dist}(0, \partial T(\Omega))$, the radius of the largest disk about the origin in the image $T(\Omega)$. Set

$$r_\Omega(a) = \sup\{r(T) : T \in \mathcal{N}(a)\}.$$

The quantity $r_\Omega(a)$ is called the *Möbius radius* of Ω at a.

It is not difficult to see that

$$r_\Omega(a) = (\min\left\{T'(0)^{-1} : T \in \mathcal{M}(a,\Omega)\right\})^{-1},$$

which yields the following interpretation of the K-P metric.

Theorem 7.2. *If Ω is a region on $\mathbb{P}$ with at least two boundary points and $a \in \Omega$, then $\mu_\Omega(a) = 2/r_\Omega(a)$. In fact, there is a unique $S \in \mathcal{N}(a)$ with $r(S) = r_\Omega(a)$.*

§8 Open problems

We close by mentioning two open questions for the K-P metric. First, determine all isometries of the K-P metric. A holomorphic function $f : \Omega \to \Delta$ is an isometry of the K-P metric if $f^*(\mu_\Delta(z)|dz|) = \mu_\Omega(z)|dz|$. Of course, Möbius transformations are isometries. It is plausible that any isometry is a Möbius transformation. We can establish this in some special situations. Second, determine the geodesics for the K-P metric in a hyperbolic region.

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