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MAPPINGS OF FINITE DISTORTION: GAUGE DIMENSION OF GENERALIZED QUASICIRCLES

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ABSTRACT. We determine the correct dimension gauge for measuring generalized quasicircles (the images of a circle under so-called μ-homeomorphisms). We establish a sharp modulus of continuity estimate for the inverse of a homeomorphism with finite exponentially integrable distortion. We exhibit several illustrative examples.

1. Introduction

We continue the study of mappings $f : \Omega \to \mathbf{R}^n$ with *finite distortion*. Thus Ω is a domain in $\mathbf{R}^n$ $(n \geq 2)$, f belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(\Omega, \mathbf{R}^n)$, the Jacobian determinant $J_f = \det(Df)$ of f (Df being the differential of f) is locally integrable in Ω, and there is a measurable function $K \geq 1$ with K finite almost everywhere and such that f satisfies the distortion inequality

$$|Df(x)|^n \leq K(x)J_f(x) \qquad \text{for almost every } x \text{ in } \Omega.$$

We also assume that f is a homeomorphism; thus, if we were to require that K be (essentially) bounded, then f would be a quasiconformal mapping (according to the analytic definition [Väi71, 34.4]). Instead of asking that K be bounded, we require only that K be exponentially integrable: there should exist a $\lambda > 0$ such that $\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega)$. A number of recent papers have established many important properties for (possibly non-homeomorphic) mappings with finite exponentially integrable distortion including: continuity, Lusin's condition (N), preservation of Lebesgue null sets, etc. See [AIKM00], [IKMS03], [IKO01], [KM03], [IM01] and the references mentioned therein.

In this article we investigate the 'size' of generalized quasispheres—these are the images $f(\mathbf{S}^{n-1})$ of the unit sphere under a homeomorphism $f : \mathbf{R}^n \to \mathbf{R}^n$ which has finite exponentially integrable distortion. In contrast to the quasiconformal case, it is not difficult to see that a generalized quasisphere

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may have Hausdorff dimension n. Thus our interest is in determining the correct dimension gauge for measuring $f(\mathbf{S}^{n-1})$. We consider dimension gauges of the form $\delta(t) = t^n(\log 1/t)^p$ with $p > 0$; we let $\Lambda^p = \Lambda^{n,p}$ denote the generalized Hausdorff measure obtained by using such a dimension function δ. (See Section 2 for basic definitions and terminology.)

While we do have some general results, we can only give a complete description for the plane case (dimension $n = 2$).

Theorem A. *There exists an absolute constant k_1 such that for any homeomorphism $f : \mathbf{R}^2 \to \mathbf{R}^2$ with finite distortion K and $\exp(\lambda K)$ locally integrable for some $\lambda > 0$, we have $\Lambda^p(f\mathbf{S}^1) < \infty$ for all $p < k_1\lambda$.*

The sharpness of the linear relationship between p and λ displayed above is a consequence of the following example.

Example A. There is an absolute constant k_2 such that for any $\lambda > 0$, there exists a homeomorphism $f : \mathbf{R}^2 \to \mathbf{R}^2$ with finite distortion K, $\exp(\lambda K)$ locally integrable, and $\Lambda^p(f\mathbf{S}^1) = \infty$ for all $p > k_2\lambda$.

This naturally leads to the following problem.

Problem A. Determine the largest constant k such that for all homeomorphisms $f : \mathbf{R}^2 \to \mathbf{R}^2$, with finite distortion K and $\exp(\lambda K)$ locally integrable, $\Lambda^p(f\mathbf{S}^1) < \infty$ for all $p < k\lambda$.

The construction of the homeomorphism f in Example A is obtained by first mapping a standard Cantor dust onto a generalized Cantor dust, and this works in $\mathbf{R}^n$. Our proof of the finiteness of the measure of a generalized quasisphere/quasicircle involves two tools: a volume distortion estimate, and the following modulus of continuity estimate.

Theorem B. *Let $\Omega \subset \mathbf{R}^n$ be a domain. Suppose that a homeomorphism $f : \Omega \to \mathbf{R}^n$ has finite distortion K with $\exp(\lambda K)$ locally integrable for some $\lambda > 0$. Fix a point $z \in \Omega$ and $0 < R < \operatorname{dist}(z, \partial\Omega)$. Then for all $|x - z| \le R/6$,*

$$|f(x) - f(z)| \ge D \exp\left(-\frac{C}{\lambda^{1/(n-1)}} \log^{n/(n-1)} \frac{\Lambda R}{|x - z|}\right),$$

where $C = C(n)$, $D = (1/2)\operatorname{dist}(fz, \partial f B(z; R/3))$, and

$$\Lambda = \left(\fint_{B(z;R)} \exp(\lambda K)\right)^{1/n}.$$

Notice that the above inequality scales appropriately when we make either change of variable $g(x) = f(\sigma x)$ or $g(x) = \sigma f(x)$. The following example illustrates that our modulus of continuity estimate is of the correct order.

EXAMPLE B. Let $\lambda > 0$ be fixed and define

$$f(x) = \rho(|x|)\frac{x}{|x|}, \qquad \text{where} \quad \rho(t) = \exp\left(-\frac{C}{\lambda^{1/(n-1)}}\log^{n/(n-1)}\frac{1}{t}\right).$$

Then f has finite distortion K with $\exp(\lambda K)$ locally integrable provided $C < (n-1)/n^{(n-2)/(n-1)}$.

Again we are led to a natural question.

PROBLEM B. Determine the smallest constant C so that the conclusion of Theorem B remains valid with this constant.

The sharpness in Theorem A relies on David's area distortion for μ-homeomorphisms; see Fact 2.3. As indicated above, there is an $\mathbf{R}^n$ version of Theorem A (in fact, the same proof works), and there exists an $\mathbf{R}^n$ analog of Example A as well. However, when $n \geq 3$ there is a lack of information regarding the precise volume distortion permitted by finite (exponentially integrable) distortion maps.

PROBLEM C. Determine whether or not the volume distortion Fact 2.3 remains valid for small values of λ. Also, determine whether or not one can take $\lambda(s;n)$ to be a linear function of s. (Here we assume $n \geq 3$.)

This paper is organized as follows. Section 2 contains preliminary information including basic definitions and terminology descriptions as well as elementary and/or well-known facts. In Section 3 we establish Theorem B and present Example B. Section 4 is devoted to corroborating Theorem A. We conclude with Section 5, where we explain and verify Example A.

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2. Preliminaries

2.A. General information. Our notation is relatively standard and conforms with that of [Väi71]. Throughout this paper Ω is a domain in Euclidean space $\mathbf{R}^n$ with $n \geq 2$ and $|A|$ denotes the n-dimensional Lebesgue measure of a measurable set $A \subset \mathbf{R}^n$. We let $u_A = \fint_A u = (1/|A|)\int_A u$ denote the integral average of a (locally integrable) function u over A. We write $B(x;r) = \{y : |x-y| < r\}$ (resp., $S(x;r) = \partial B(x;r) = \{y : |x-y| = r\}$) for the open ball (sphere) of radius r centered at the point x. We put $\mathbf{B}^n = B(0;1)$ (the unit ball), $\mathbf{S}^{n-1} = S(0;1)$ (the unit sphere), $\Omega_n = |\mathbf{B}^n|$ and let ω_{n-1} denote the measure of $\mathbf{S}^{n-1}$. We write $C = C(a,\dots)$ to indicate a constant C which depends only on the parameters $a,\dots$; $a \lesssim b$ means there exists a positive finite constant C with $a \leq C\,b$, and $a \simeq b$ means $a \lesssim b \lesssim a$. Typically c, C will be constants (whose actual value may vary even in the same line) which depend on various parameters, and we try to make this as clear as possible, often giving explicit values. We reserve the letter k for an absolute constant.

Recall that the *generalized Hausdorff measure*, $\mathcal{G}^\delta$, is given by

$$\mathcal{G}^\delta(A) = \lim_{r\to 0}\left[\inf\left\{\sum \delta(\operatorname{diam} U_i) : A \subset \bigcup U_i, \operatorname{diam}(U_i) \le r\right\}\right],$$

where δ is a dimension gauge (non-decreasing with $\delta(0) = 0$). When $\delta(t) = t^\alpha$ for some $\alpha \ge 0$, we write $\mathcal{H}^\alpha = \mathcal{G}^\delta$ and call this the *Hausdorff α-dimensional measure*. When $\delta(t) = t^n(\log 1/t)^p$ with $p > 0$, we let $\Lambda^p = \Lambda^{n,p} = \mathcal{G}^\delta$. For the most part, $\mathcal{H}^\alpha$ and Λ^p will be the only Hausdorff measures of interest to us. Mattila's book [Mat95] is an excellent reference for this material.

Next we state a result which describes the integrability of the Jacobian for certain mappings. This follows from the area formula in conjunction with the fact that such mappings are essentially Lipschitz on large subsets; see [IKM02, Thm 6.1], [IM01, Cor 6.3.1] and/or [Fed69, 3.1.8], [EG92, §§6.1.3,6.6.3].

2.1. Fact. *Suppose a homeomorphism f belongs to the Sobolev space $W^{1,1}_{\mathrm{loc}}(B, \mathbf{R}^n)$, where $B \subset \mathbf{R}^n$ is a ball. Then for each non-negative measurable function $u : \mathbf{R}^n \to \mathbf{R}$,*

$$\int_A u \circ f |J_f| \le \int_{\mathbf{R}^n} u \qquad \text{for all measurable } A \subset B.$$

2.B. Mappings of finite distortion. Here we collect some information about mappings of finite distortion.

2.2. Lemma. *Let g, h be self-homeomorphisms of $\mathbf{R}^n$, one of which is quasiconformal and the other having finite exponentially integrable distortion. Then $f = h \circ g$ also has finite exponentially integrable distortion.*

Proof. Let K_g, K_h denote distortion functions for g, h, respectively. When h is quasiconformal (so that K_h is essentially bounded), it is straightforward to see that f has finite distortion $K_f \le \|K_h\|_\infty K_g$, and also that $\exp(\lambda K_f)$ is locally integrable for $\lambda \le \mu/\|K_h\|_\infty$ whenever $\exp(\mu K_g)$ is locally integrable.

Assume that g is the quasiconformal homeomorphism (so K_g is essentially bounded). Again, f has finite distortion $K_f \le \|K_g\|_\infty K_h \circ g$. To check the exponential integrability, we use Hölder's inequality to obtain

$$\int_B \exp(\lambda K_f) \le \left(\int_B \exp(n\lambda K_f/s)|Dg|^n\right)^{s/n}\left(\int_B |Dg|^{ns/(s-n)}\right)^{(n-s)/n},$$

which is valid for any $0 < s \le n$, with B any ball. The obvious change of variables shows that the first integral on the right-hand side above is finite, provided we choose $\lambda \le (s/n)(\mu/\|K_g\|_\infty)$ and $\exp(\mu K_h)$ is locally integrable.

To handle the second integral on the right-hand side above, we estimate

$$1/|Dg|^{ns/(n-s)} \le 1/J_g^{s/(n-s)} = (J_{g^{-1}} \circ g)^{s/n-s},$$

which, together with a similar change of variables, gives us

$$\int_B |Dg|^{ns/(s-n)} \leq \int_{g(B)} J_{g^{-1}}^{n/(n-s)}.$$

According to Gehring's lemma concerning higher integrability for quasiconformal maps ([Geh73], [IM01, 14.0.2]), there is an exponent $p = p(n, \|K_g\|_\infty) > n$ such that $J_{g^{-1}}^{p/n}$ is locally integrable. Choosing $s = n(p-n)/p = n(1-n/p)$ (so $0 < s < n$) we obtain $n/(n-s) = p/n$ and thus the above integral is finite as desired.

Note that in the second case above, we demonstrated that $\exp(\lambda K_f)$ is locally integrable when $\lambda \leq (1-n/p)\mu/\|K_g\|_\infty$ and $\exp(\mu K_h)$ is locally integrable, where $p = p(n, \|K_g\|_\infty) > n$ is Gehring's higher integrability exponent (for the inverse of the quasiconformal map g). $\square$

The following result gives a volume distortion estimate for mappings with finite distortion; see [IKMS03, Cor 2], [AIKM00, Thm 6.1]. The improved version for $n = 2$ is a consequence of David's work [Dav88], and this is what gives us the sharpness in Theorem A.

2.3. FACT. *For each $n \geq 2$ and $s \geq 0$ there is a constant $\lambda(s; n) \geq 1$ such that for $\lambda \geq \lambda(s; n)$ we have*

$$|f(E)| \leq C/\log^s\left(2 + \frac{1}{|E|}\right) \qquad \textit{for all compact } E \subset \mathbf{B}^n$$

whenever $f : B(0;2) \to \mathbf{R}^n$ has finite distortion K with $\exp(\lambda K)$ locally integrable in $B(0;2)$. Here C is a constant which depends only on f, s, n.

When $n = 2$ and f is a homeomorphism, we can take $\lambda(s; 2) = s/k_0$, where k_0 is some absolute constant.

2.C. Capacity. The *(conformal) capacity* of a compact set $E \subset \Omega$, relative to Ω, is

$$\operatorname{cap}(E; \Omega) = \inf_{\mathcal{W}} \int_\Omega |\nabla u|^n,$$

where $\mathcal{W}$ is the family of all functions u which are continuous in Ω, possess weak derivatives whose nth powers are integrable, have zero 'boundary values', and satisfy $u \geq 1$ on E. Standard arguments permit us to assume that $u \in \mathcal{C}_0^\infty(\Omega)$ with $0 \leq u \leq 1$, and we call these latter functions *admissible* for $\operatorname{cap}(E; \Omega)$; see [HKM93, pp. 27–28].

Here is a result which can be proved using the Sobolev embedding theorem on spheres, or alternatively by adapting the proof of [HK98, Theorem 5.9]. For the reader's convenience, we outline the former argument.

2.4. LEMMA. *Let E be a continuum joining the origin to the unit sphere. Suppose that $u \in W^{1,1}(\mathbf{B}^n, \mathbf{R})$ is continuous and satisfies: $u \geq 0$, $u \geq 1$ on*

E, and $u_{\mathbf{B}^n} \leq 1/2$. Then there is a constant $C(p,n) > 0$ such that for all $n-1 < p < n$,

$$\int_{\mathbf{B}^n} |\nabla u|^p \geq C(p,n).$$

Proof. First, the hypotheses on u guarantee that the set T of $t \in (0,1)$ with $\max_{|x|=t} u - \min_{|x|=t} u \geq 1/4$ has measure $|T| \geq 1 - (2/3)^{1/n}$. Indeed,

$$\frac{1}{2} \geq u_{\mathbf{B}^n} = \frac{1}{\Omega_n} \int_{\mathbf{S}^{n-1}} \int_0^1 u(r\omega) r^{n-1}\, dr\, d\omega \geq \frac{3}{4} \frac{\omega_{n-1}}{\Omega_n} \int_{[0,1]\setminus T} r^{n-1}\, dr,$$

so

$$|[0,1] \setminus T|^n \leq n \int_{[0,1]\setminus T} r^{n-1}\, dr \leq \frac{2}{3}.$$

Fix $n-1 < p < n$ and assume that $u \in W^{1,p}(\mathbf{B}^n)$. Then by Fubini's theorem, $u \in W^{1,p}(S(t))$ for a.e. $t \in (0,1)$. The Sobolev embedding theorem now permits us to assert that for a.e. $t \in (0,1)$ and all $x, y \in S(t)$,

$$|u(x) - u(y)| \leq C_1(p,n)\, t \left(\fint_{S(t)} |\nabla u|^p \right)^{1/p}.$$

Thus we have

$$\int_{S(t)} |\nabla u|^p t^{n-1}\, d\omega \geq C_2(p,n)\, t^{n-p-1} \qquad \text{for a.e. } t \in T,$$

which yields the desired conclusion

$$\int_{\mathbf{B}^n} |\nabla u|^p \geq C_2(p,n) \int_T t^{n-p-1}\, dt \geq C(p,n)$$

with $C(p,n) = C_2/(n-p)[1 - (2/3)^{1-p/n}]$ and $C_2 = \omega_{n-1}/[4C_1]^p$. □

2.5. COROLLARY. *Let E be a continuum joining some point a to the sphere $S(a;r)$. Suppose that $v \in W^{1,1}(B(a;r), \mathbf{R})$ is continuous and satisfies $v \geq 1$ on E and $v_{B(a;r)} \leq 1/2$. Then for all $n-1 < p < n$,*

$$\fint_{B(a;r)} |\nabla v|^p \geq C(p,n)\, r^{-p}.$$

Proof. Apply Lemma 2.4 to $u(x) = v(a + rx)$. □

3. Modulus of continuity of f^{-1}

In this section we establish Theorem B and present Example B.

3.A. Proof of Theorem B. Let $f : \Omega \to \mathbf{R}^n$ be a homeomorphism with finite distortion K and $\exp(\lambda K) \in L^1_{\mathrm{loc}}(\Omega)$ for some $\lambda > 0$. Fix a point $z \in \Omega$ and $0 < R < \operatorname{dist}(z, \partial\Omega)$. By considering the change of variables $x = z + (2/3)R\xi$, we may assume that $z = 0$ and $R = 3/2$. We may further assume that $f(0) = 0$. We now have $D = (1/2)\operatorname{dist}(0, \partial B')$, where $B' = f(B)$, $B = B(0; 1/2)$, and we must demonstrate that for all $|x| \le 1/4$,

$$|f(x)| \ge D \exp\left(-\frac{C}{\lambda^{1/(n-1)}} \log^{n/(n-1)} \frac{3\Lambda}{2|x|}\right),$$

with $C = C(n)$ and $\Lambda = (⨍_{B(0;3/2)} \exp(\lambda K))^{1/n}$.

Fix a point a with $|a| \le 1/4$. We may assume that $a' = f(a)$ satisfies $|a'| < D$, for otherwise we are done. Then $\operatorname{dist}(a', \partial B') > D$, so the line segment $E' = [0, a']$ lies in B' and we have the capacity estimate

$$\operatorname{cap}(E'; B') \le \omega_{n-1}/(\log D/|a'|)^{n-1}. \tag{3.1}$$

Note that if f were quasiconformal, then we could now finish our argument by appealing to a capacity estimate in the original domain.

Let u be an admissible function for $\operatorname{cap}(E'; B')$, so u is smooth with support in B', $u \ge 0$, and $u \ge 1$ on E'. Put $v = u \circ f$; since f belongs to $W^{1,1}(B', \mathbf{R}^n)$, we know that ∇v exists a.e., $|\nabla v| = |Df||\nabla u \circ f|$ is locally integrable, and $v \in W^{1,1}(B, \mathbf{R})$. Since f has finite exponentially integrable distortion, we can use the distortion inequality together with Fact 2.1 to obtain the estimate

$$\int_B \frac{|\nabla v|^n}{K} \le \int_B |\nabla u \circ f|^n J_f \le \int_{B'} |\nabla u|^n. \tag{3.2}$$

Our final task is to establish the lower bound

$$\int_B \frac{|\nabla v|^n}{K} \ge C(n)\lambda \left(\log \frac{3\Lambda}{2|a|}\right)^{-n}. \tag{3.3}$$

Before we get into too many details, let us see why this finishes the proof. Indeed, combining (3.3) with (3.2), then taking an infimum over all admissible functions u and employing (3.1), we conclude that

$$C(n)\lambda \left(\log \frac{3\Lambda}{2|a|}\right)^{-n} \le \omega_{n-1} \left(\log \frac{D}{|a'|}\right)^{1-n},$$

and therefore

$$\log \frac{D}{|a'|} \le \frac{C(n)}{\lambda^{1/(n-1)}} \log^{n/(n-1)} \frac{3\Lambda}{2|a|},$$

which gives the desired inequality. (Note that $3\Lambda/2|a| \ge e$.)

To validate (3.3), we consider two cases depending on whether or not the average value of v over the ball $A = B(0; |a|)$ exceeds $1/2$. Define

$$L = \int_{B(0;3/2)} \exp(\lambda K), \qquad \text{so} \qquad \frac{L}{|A|} = \left(\frac{3\Lambda}{2|a|}\right)^n.$$

The case $v_A \le 1/2$. Here we can appeal to Corollary 2.5 and assert that

$$\fint_A |\nabla v|^p \ge c\, r^{-p} \qquad \text{for any } n-1 < p < n,$$

where $r = |a|$ and $c = c(p,n)$. Take $p = n^2/(n+1)$ and apply Hölder's inequality to obtain

$$\frac{c}{r^p} \le \left(\fint_A \frac{|\nabla v|^n}{K}\right)^{p/n} \left(\fint_A K^n\right)^{(n-p)/n}. \tag{3.4}$$

Our next goal is to obtain an upper bound for $\fint_A K^n$. Consider the auxiliary function

$$\varphi(t) = \exp(\lambda t^{1/n}).$$

Certainly φ is increasing, and an easy calculation shows that $\varphi(t)/t$ is also increasing for all $t \ge (n/\lambda)^n$. As $3\Lambda/2|a| \ge e$, we get

$$\tau = \left(\frac{1}{\lambda}\log\frac{L}{|A|}\right)^n \ge \left(\frac{n}{\lambda}\right)^n.$$

Thus $\varphi(t)$ and $\varphi(t)/t$ are increasing for all $t \ge \tau$ and we deduce that

$$\begin{aligned}\int_A K^n &\le \int_{\{K^n \ge \tau\}} K^n + \tau|A| \le \frac{\tau}{\varphi(\tau)} \int_{\{K^n \ge \tau\}} \varphi(K^n) + \tau|A| \\ &\le \frac{\tau}{\varphi(\tau)} \int_{B(0;3/2)} \exp(\lambda K) + \tau|A| = 2\tau|A|;\end{aligned}$$

the equality just above is a consequence of the fact that $\varphi(\tau) = L/|A|$. This estimate can be written as

$$\left(\fint_A K^n\right)^{(n-p)/n} \le C(n)\left(\frac{1}{\lambda}\log\frac{L}{|A|}\right)^{p/n}.$$

Employing this inequality in (3.4) yields

$$\fint_A \frac{|\nabla v|^n}{K} \ge C(n)\lambda/\log\frac{L}{|A|},$$

which in turns gives (3.3) (because $3\Lambda/2|a| \ge e$); in fact, here we obtain a stronger inequality since there is no exponent n on the logarithm term.

The case $v_A \ge 1/2$. Here we utilize a chaining argument together with a Poincaré inequality. In order to facilitate a technical calculation below, we now rescale to get "$L/\Omega_n = 1$". Consider the change of variable $g(x) = f(x/\sigma)$. Then $K_g(x) = K(x/\sigma)$ is a finite distortion function for g with

$$L_g = \int_{B(0;3\sigma/2)} \exp(\lambda K_g) = \sigma^n L,$$

so taking $\sigma = (\Omega_n/L)^{1/n}$ we obtain $L_g = \Omega_n$. Next, let $w(x) = v(x/\sigma)$ and note that $w_{\sigma A} = v_A$ and also $\int_{\sigma B} |\nabla w|^n/K_g = \int_B |\nabla v|^n/K$. Thus we are still in the case $w_{\sigma A} \geq 1/2$ searching for a lower bound for the integral $\int_{\sigma B} |\nabla w|^n / K_g$.

Let m be the positive integer with $1/2^{m+1} < \sigma|a| \leq 1/2^m$. Then $m \geq 2$. Let $b = (1, 0, \ldots, 0)$ and consider the balls $A_i = B(a_i; r_i/2)$, $B_i = B(b_i; r_i)$, where $r_i = 1/2^{m-i+1}$, $b_i = 2r_i b$, and $a_i = b_i + (r_i/2)b$; here $i \geq 1$. Also, put $B_0 = B(0; 1/2^m)$ and $A_0 = B(a_0; r_1/4)$, where $a_0 = (5/4)r_1 b$. Then for $i \geq 1$ we have $A_{i-1}, A_i \subset B_i$ with each of ∂A_{i-1}, ∂A_i being tangent to ∂B_i and $2 \operatorname{diam} A_{i-1} = \operatorname{diam} A_i = (1/2) \operatorname{diam} B_i$; also, $\sigma A, A_0 \subset B_0$.

Let ℓ be the smallest integer with $1/2^{m-\ell} \geq \sigma/2$; so $1 \leq \ell \leq m-1$ as $\sigma < 1$. Then A_ℓ lies in the complement of $\sigma B = B(0; \sigma/2)$, so $w_{A_\ell} = 0$ because the support of w lies in σB. Thus we can write

$$1/2 \leq w_{\sigma A} = (w_{\sigma A} - w_{A_0}) + (w_{A_0} - w_{A_1}) + \cdots + (w_{A_{\ell-1}} - w_{A_\ell}).$$

Next, employing a Poincaré inequality, we can estimate the absolute value of each of these terms, thereby obtaining

$$C(n) \leq \sum_{i=0}^{\ell} \operatorname{diam}(B_i) \fint_{B_i} |\nabla w|.$$

Now we use Hölder's inequality twice, first on each of the integrals, and then on the sum itself, to get

$$C(n) \leq \left(\sum_{i=0}^{\ell} (\operatorname{diam}(B_i))^n \fint_{B_i} \frac{|\nabla w|^n}{K_g} \right)^{1/n} \left(\sum_{i=0}^{\ell} \fint_{B_i} K_g^{1/(n-1)} \right)^{(n-1)/n}.$$

The first factor on the right-hand side above can be estimated from above by (a constant times) $(\int_{\sigma B} |\nabla w|^n / K_g)^{1/n}$; this is because $B_i \cap \operatorname{supp}(w) \subset \sigma B$ and the balls B_i have bounded overlap. Thus, raising to the power n provides us with

$$C(n) \leq \left(\int_{\sigma B} \frac{|\nabla w|^n}{K_g} \right) \left(\sum_{i=0}^{\ell} \fint_{B_i} K_g^{1/(n-1)} \right)^{n-1}. \tag{3.5}$$

It therefore remains to exhibit an upper bound for $\left(\sum \fint_{B_i} K_g^{1/(n-1)} \right)^{n-1}$.

In fact, we verify that

$$\left(\sum_{i=0}^{\ell} \fint_{B_i} K_g^{1/(n-1)} \right)^{n-1} \leq \frac{C(n)}{\lambda} m^n. \tag{3.6}$$

Then, since $m \simeq \log(1/\sigma|a|)$, and recalling the definitions of σ and L, we see that (3.3) is an immediate consequence of (3.6) in conjunction with (3.5).

Notice that $B_i \subset (3\sigma/2)\mathbf{B}^n$ for $0 \le i \le \ell$. Thus an application of Jensen's inequality, with the auxiliary function $\varphi(t) = \exp(\lambda t^{n-1})$, yields

$$\begin{aligned}\varphi\left(\fint_{B_i} K_g^{1/(n-1)}\right) &\le \fint_{B_i} \exp(\lambda K_g) \le \frac{1}{|B_i|}\int_{B(0;3\sigma/2)} \exp(\lambda K_g)\\ &= \frac{L_g}{|B_i|} = r_i^{-n};\end{aligned}$$

recall the rescaling done above to ensure that $L_g = \Omega_n$. Since $\varphi^{-1}(s) = [(1/\lambda)\log s]^{1/(n-1)}$ and $r_i = 1/2^{m-i+1}$, we deduce that

$$\fint_{B_i} K_g^{1/(n-1)} \le \left(\frac{1}{\lambda}\log\frac{1}{r_i^n}\right)^{1/(n-1)} = C(n)\lambda^{1/(1-n)}(m-i+1)^{1/(n-1)}.$$

Now as i runs through the indices $0, 1, \ldots, \ell$ we have $m - i + 1$ taking on the values $m+1, m, \ldots, m-\ell+1$. Since $m - \ell + 1 \ge 2$, we have

$$\sum_{i=0}^{\ell}(m-i+1)^{1/(n-1)} = \sum_{j=m-\ell+1}^{m+1} j^{1/(n-1)} \le \sum_{j=1}^{m+1} j^{1/(n-1)} \le C(n)m^{n/(n-1)}.$$

Combining the two inequalities displayed just above corroborates (3.6), thereby completing our proof. □

3.B. Demonstration of Example B. Let

$$f(x) = \rho(|x|)\frac{x}{|x|}, \qquad \text{where} \quad \rho(t) = \exp\left(-\frac{C}{\lambda^{1/(n-1)}}\log^{n/(n-1)}\frac{1}{t}\right);$$

here $\lambda > 0$ is a fixed parameter and $C = C(n)$ a constant to be determined. Since f is radial, there are only two directional derivatives to check, and we conclude that

$$|Df(x)| = \max\{\rho'(r), \rho(r)/r\} \qquad \text{and} \qquad J_f(x) = \rho'(r)\left(\rho(r)/r\right)^{n-1},$$

where $r = |x|$. As

$$\rho'(t) = \frac{C}{\lambda^{1/(n-1)}}\frac{n}{n-1}\log^{1/(n-1)}\frac{1}{t}\frac{\rho(t)}{t},$$

we find that $|Df(x)| = \rho'(r)$, at least for $r = |x| << 1$. Then

$$K(x) = \frac{|Df(x)|^n}{J_f(x)} = \left(\frac{r\rho'(r)}{\rho(r)}\right)^{n-1} = [(Cn/(n-1))^{n-1}/\lambda]\log\frac{1}{r}.$$

We conclude that $\exp(\lambda K) = \exp(\alpha\log 1/r) = r^{-\alpha}$, where $\alpha = (Cn/(n-1))^{n-1}$. Thus $\exp(\lambda K)$ will be locally integrable precisely when $C < C(n) = (n-1)n^{-(n-2)/(n-1)}$. □

We point out that the constant $C(n) = (n-1)n^{-(n-2)/(n-1)}$ gives a lower bound for Problem B.

4. Generalized quasicircles

In this section we establish Theorem A and state a few observations concerning its proof and higher dimensional analogs.

4.A. Proof of Theorem A. Suppose $f : \mathbf{R}^2 \to \mathbf{R}^2$ is a homeomorphism with finite distortion K and $\exp(\lambda K)$ locally integrable for some $\lambda > 0$. Assume that $f(0) = 0$ and choose $R > 6$ so that $\operatorname{dist}(fz, \partial f B(z; R/3)) \geq 2$ for all $z \in \mathbf{S}^1$. Consider any cover of $f(\mathbf{S}^1)$ by disks $B(fz_i; r)$, where $z_i \in \mathbf{S}^1$ and $r > 0$ is small. Since f^{-1} is uniformly continuous (on $f(\mathbf{S}^1)$)), we can assume that $f^{-1}B(fz_i; r) \subset B(z_i; 1)$ for all z_i. Appealing to Theorem B we find that $f(x) \in B(fz_i; r)$ implies

$$r \geq |f(x) - f(z_i)| \geq \exp\left(-\frac{C}{\lambda} \log^2 \frac{\Lambda R}{|x - z_i|}\right),$$

and therefore $|x - z_i| \leq \Lambda R \exp(-[(\lambda/C)\log(1/r)]^{1/2})$. In particular, we see that $\bigcup B(fz_i; r) \subset f(A_\varepsilon)$, where

$$A_\varepsilon = \{x : 1 - \varepsilon < |x| < 1 + \varepsilon\} \qquad \text{and} \qquad \log^2 \frac{\Lambda R}{\varepsilon} = \frac{\lambda}{C} \log \frac{1}{r}.$$

Let $N(E, r)$ denote the maximal number of disjoint disks with centers in E and radii r. Then the above, in conjunction with Fact 2.3, yields

$$N(f\mathbf{S}^1, r)\pi r^2 \leq |f(A_\varepsilon)| \leq C \log^{-s} \frac{1}{|A_\varepsilon|} = C \log^{-s} \frac{1}{4\pi\varepsilon}$$

provided $s < k_0\lambda$. Thus for $p \leq s/2 < k_0\lambda/2$ we find that

$$\limsup_{r \to 0} N(f\mathbf{S}^1, r)\, r^2 \log^p \frac{1}{r} < \infty,$$

which in particular gives $\Lambda^p(f\mathbf{S}^1) < \infty$ (cf. [Mat95, pp. 76–79]). $\square$

4.B. Remarks. (1) The above proof certainly works in $\mathbf{R}^n$, but the volume distortion fact 2.3 requires that $\lambda \geq \lambda(s; n)$ be sufficiently large.

(2) The alert reader will no doubt see that our proof actually concerns generalized Minkowski content obtained by using the dimension gauge $\delta(t) = t^n(\log 1/t)^p$. We mention that in the quasiconformal category one knows that

$$\sup \dim_{\mathcal{H}} f(\mathbf{S}^1) = \sup \dim_{\mathcal{M}} f(\mathbf{S}^1) = 1 + \left(\frac{K-1}{K+1}\right)^2,$$

where the supremum is taken over all K-quasiconformal self-homeomorphisms of the plane and $\dim_{\mathcal{H}}$, $\dim_{\mathcal{M}}$ stand for Hausdorff, Minkowski dimensions, respectively. Astala [Ast88, Theorem 1.5] established the first identity and conjectured the correct dimension, which has recently been corroborated by Smirnov. It would be interesting to know whether or not one can say anything at all like this for finite distortion homeomorphisms.

(3) There are second order estimates which suggest a linear relationship between λ and p even when $n \geq 3$. Indeed, there is a volume distortion estimate of the form

$$|f(E)| \leq C/\left(\log\log\frac{1}{|E|}\right)^p \qquad \text{for all compact } E,$$

and this is valid for all $p > 0$ and all $\lambda > 0$ (see [IM01, p. 160], [AIKM00, Thm. 6.1]). Using this in the above proof leads to the conclusion that for any homeomorphism f having finite distortion K with $\exp(\lambda K)$ locally integrable for some $\lambda > 0$, $\mathcal{G}^\delta(f\mathbf{S}^{n-1}) < \infty$ for every dimension gauge $\delta(t) = t^n(\log\log(1/t))^p$ with $p > 0$.

5. Cantor dust example

Here we construct a natural self-homeomorphism of $\mathbf{R}^n$ which maps a standard Cantor dust onto a generalized one. We determine when this mapping will have exponentially integrable distortion, and then utilize this to corroborate Example A.

5.A. From dust to dust. We recall one of the ways to construct a 'standard' Cantor dust. Starting with the unit interval $[0,1]$, we select two intervals, each of length σ, one from the middle of $[0,1/2]$ and one from the middle of $[1/2,1]$; here $0 < \sigma < 1/2$ is some fixed parameter. This gives us our first generation basic intervals. We now iterate this process: given a $(k-1)$st generation basic interval I, we select two intervals each of length σ^k, one from the middle of each of the two halves of I. This will give us a total of 2^k kth generation intervals each of length σ^k. Then $\mathcal{C} = \mathcal{C}(\sigma)$ is the intersection over all generations of the unions of all kth generation basic intervals. Standard arguments give the dimension of $\mathcal{C}$ as $\alpha = \log 2/\log(1/\sigma)$, and show that $\mathcal{C}$ has positive finite $\mathcal{H}^\alpha$-measure.

The above construction can be generalized by replacing the fixed parameter σ with a sequence $\{\tau_k\}$, where $0 < \tau_k < 1/2$. Here the kth generation basic intervals have length $t_k = \tau_1 \dots \tau_k$. Again, standard arguments reveal that $\mathcal{C} = \mathcal{C}(\{\tau_k\})$ has positive finite generalized Hausdorff measure $\mathcal{G}^\delta(\mathcal{C})$ provided our dimension gauge δ and parameter sequence $\{\tau_k\}$ satisfy $\delta(t_k) \simeq 2^{-k}$. In particular, if we take

$$\tau_1 = \frac{1}{2}\frac{1}{(\log 4)^p} \qquad \text{and} \qquad \tau_k = \frac{1}{2}\left(1-\frac{1}{k}\right)^p \quad \text{for } k = 2,3,\dots,$$

then we find that $\mathcal{C} = \mathcal{C}(\{\tau_k\})$ has positive finite generalized Hausdorff measure $\mathcal{G}^\delta(\mathcal{C})$ for the gauge $\delta(t) = t(\log(1/t))^p$, where $p > 0$. Since $t < t(\log(1/t))^p < t^\alpha$ as $t \to 0$ (for any fixed $0 < \alpha < 1$), such a generalized Cantor dust is 'bigger' than any standard Cantor dust.

We remark that in either of the above constructions we can form $\mathcal{C}^n = \mathcal{C} \times \cdots \times \mathcal{C}$ and obtain a Cantor dust in $\mathbf{R}^n$, and this set has positive finite measure, either in dimension $n \log 2/\log(1/\sigma)$ for a 'standard' dust, or using the gauge $\delta(t) = t^n(\log(1/t))^{pn}$ for a generalized dust. We are now ready to construct a self-homeomorphism of $\mathbf{R}^n$, which maps such a 'standard' dust onto a generalized one, and has finite exponentially integrable distortion provided the associated parameters satisfy a certain inequality. Our method here has been utilized in [KKM01b, §4], [KKM01a, §5], [IM01, §6.5.6], as well as other places.

5.1. PROPOSITION. *Let $\mathcal{C} = \mathcal{C}(\sigma)^n$, $\mathcal{C}' = \mathcal{C}(\{\tau_k\})^n$ be 'standard', resp. generalized Cantor dusts in $\mathbf{R}^n$ constructed (as above) using parameters $0 < \sigma < 1/2$, resp. $0 < \tau_k < 1/2$. There is a finite distortion homeomorphism $h : \mathbf{R}^n \to \mathbf{R}^n$ with the property that $h(\mathcal{C}) = \mathcal{C}'$. Moreover, if we take*

$$\tau_1 = \frac{1}{2}\frac{1}{(\log 4)^p} \qquad \textit{and} \qquad \tau_k = \frac{1}{2}\left(1 - \frac{1}{k}\right)^p \quad \textit{for } k = 2, 3, \ldots,$$

for some $p > 0$, then the distortion function K_h for h will have $\exp(\mu K_h)$ locally integrable provided $\mu < C(\sigma)pn$, where $C(\sigma) = \log(1/2\sigma)/((1/2\sigma)-1)$.

Proof. Fix $0 < \sigma < 1/2$, $0 < \tau_k < 1/2$ and let $\mathcal{C} = \mathcal{C}(\sigma)^n$, $\mathcal{C}' = \mathcal{C}(\{\tau_k\})^n$ be the Cantor dusts in $\mathbf{R}^n$ as described above. There are 2^{kn} cubes Q_{ki}, of edge length $2r_k = \sigma^k$ associated with each of the 2^k kth generation basic intervals for $\mathcal{C}$, and similar cubes Q'_{ki}, of edge length $2r'_k = t_k = \tau_1 \ldots \tau_k$ for $\mathcal{C}'$. There are also larger cubes P_{ki} (concentric about Q_{ki}), of edge length $2R_k = (1/2)\sigma^{k-1}$, and P'_{ki} (concentric about Q'_{ki}), of edge length $2R'_k = (1/2)t_{k-1} = (1/2)\tau_1 \ldots \tau_{k-1}$. (Thus Q_{ki}, Q'_{ki} are formed using the kth generation basic intervals while P_{ki}, P'_{ki} are formed using the halves of the $(k-1)$st generation basic intervals.) Next we put $A_{ki} = P_{ki} \setminus Q_{ki}$, $A'_{ki} = P'_{ki} \setminus Q'_{ki}$; these 'cubical collars' (or 'frames') are actually spherical rings when we work with the norm

$$\|x\| = \|x\|_\infty = \max\{|x_1|, \ldots, |x_n|\},$$

because the cubes $P_{ki}, Q_{ki}, P'_{ki}, Q'_{ki}$ are all just balls in this norm.

Now we describe how to produce a 'natural' self-homeomorphism of $\mathbf{R}^n$ which maps $\mathcal{C}$ onto $\mathcal{C}'$. We start by defining $h_1 : \mathbf{R}^n \to \mathbf{R}^n$ as

$$h_1 = \begin{cases} \text{the identity outside all } P_{1i}, \\ \text{a similarity of } Q_{1i} \text{ onto } Q'_{1i}, \\ \text{a 'radial stretching' of } A_{1i} \text{ onto } A'_{1i}. \end{cases}$$

(The patient reader will see that we explain this 'radial stretching' in more detail below.) Note that the only distortion for h_1 is that from the 'radial stretching' and this all 'lives' in the cubical collars A_{1i}. We iterate this construction by defining $h_k : \mathbf{R}^n \to \mathbf{R}^n$ via

$$h_k = \begin{cases} h_{k-1} \text{ outside all } P_{ki}, \\ \text{a similarity of } Q_{ki} \text{ onto } Q'_{ki}, \\ \text{a `radial stretching' of } A_{ki} \text{ onto } A'_{ki}. \end{cases}$$

(The reader interested in more precise formulae for these maps is invited to consult, e.g., [KKM01b, §4] or [KKM01a, §5] or [IM01, §6.5.6].) Again, the only (new) distortion for h_k 'lives' in the collars A_{ki}. In the usual way we now obtain a self-homeomorphism $h = \lim_{k\to\infty} h_k$ of $\mathbf{R}^n$ which maps the unit cube onto itself, and satisfies $h(\mathcal{C}) = h(\mathcal{C}')$ too. Moreover, the distortion of h 'lives' in the union of all the cubical collars A_{ki}, and since these collars are all disjoint, the distortion of h in A_{ki} is just that coming from the map h_k, i.e., coming from the radial stretching of A_{ki} onto A'_{ki}.

Next we take a more careful look at these radial stretchings. Consider the cubical collars (i.e., spherical rings)

$$A = \{x : r < \|x\| < R\} \qquad \text{and} \qquad A' = \{y : r' < \|y\| < R'\}$$

and the radial homeomorphism

$$y = \varphi(x) = \frac{x}{\|x\|}\rho(\|x\|), \qquad \text{where} \quad \rho(t) = at + b.$$

Since $\|x\| = t$ maps to $\|y\| = at + b$, we see that $A' = \varphi(A)$ provided we choose a, b to satisfy

$$r' = ar + b \quad , \quad R' = aR + b \quad ; \qquad \text{so} \qquad a = \frac{R' - r'}{R - r} \quad , \quad \text{etc.}$$

Moreover, it is straightforward [KKM01b, 4.1] to obtain the estimates

$$|D\varphi(x)| \simeq a + \frac{b}{\|x\|} \qquad \text{and} \qquad J_\varphi(x) = a\left(a + \frac{b}{\|x\|}\right)^{n-1},$$

and thus φ has distortion

$$K_\varphi(x) \simeq 1 + \frac{b}{a\|x\|}.$$

Now we examine the distortion of h in A_{ki}. Recall that the cubical collar A_{ki} has inner, outer radii r_k, $R_k = \sigma^{k-1}/4 = r_k/(2\sigma)$. Thus $R_k \simeq r_k$ and so we deduce that h has distortion

$$K_h \simeq 1 + \frac{b}{ar} = \frac{r'}{r}\frac{R - r}{R' - r'} \qquad \text{in } A_{ki},$$

where $r = r_k$, $R = R_k$, $r' = r'_k$, $R' = R'_k$ and $a = a_k = (R' - r')/(R - r)$, $b = b_k$ are chosen so that A_{ki} is mapped onto A'_{ki}. Writing r, R, r', R' in terms of our parameters and simplifying we find that

$$K_h \simeq \frac{\tau_k}{\sigma}\frac{1-2\sigma}{1-2\tau_k} \qquad \text{in } A_{ki}.$$

Now we estimate the integral of $\exp(\mu K_h)$ over the unit cube, which can be found by summing over all cubical collars A_{ki}. Since there are 2^{nk} of these collars, each of measure no larger than R_k^n (the volume of P_{ki}), the above estimate for K_h yields

$$\int_{[0,1]^n} \exp(\mu K_h) \lesssim \sum_{k=1}^{\infty} 2^{nk} R_k^n \exp\left(\mu \frac{\tau_k}{\sigma}\frac{1-2\sigma}{1-2\tau_k}\right) \lesssim \sum_{k=1}^{\infty} c_k \rho^k,$$

where

$$c_k = \exp\left(\mu \frac{\tau_k}{\sigma}\frac{1-2\sigma}{1-2\tau_k}\right) \qquad \text{and} \qquad \rho = (2\sigma)^n.$$

The venerable Ratio Test assures us that the above series converges provided

$$\exp\left(\frac{\mu}{p}\frac{1-2\sigma}{2\sigma}\right) = \lim_{k\to\infty}\frac{c_{k+1}}{c_k} < \frac{1}{\rho} = (2\sigma)^{-n};$$

i.e., $\exp(\mu K_h)$ will be locally integrable if $\mu < C(\sigma)\,p\,n$. □

5.B. Demonstration of Example A. Let $\sigma = 1/4$ and construct a 'standard' Cantor set $\mathcal{C} \subset \mathbf{R}^2$ as above using this parameter. There is a quasiconformal homeomorphism $g : \mathbf{R}^2 \to \mathbf{R}^2$ with the property that $g(\mathbf{S}^1) \supset \mathcal{C}$; see [GV71], [Geh82, 3.2], [Bis99, 3.1]. Now g has (outer) dilatation (i.e., a distortion function) bounded by some absolute constant, say K_0, and appealing to Gehring's higher integrability lemma ([Geh73], [IM01, 14.0.2]) there is an exponent $p_0 = p_0(2, K_0) > 2$ such that $J_{g^{-1}}^{p_0/2}$ is locally integrable. Put $k_2 = 2p_0K_0/(p_0-2)$ (an absolute constant).

Let $\lambda > 0$ be given, and set $q = \mu = (k_2/2)\lambda$. As in Subsection 5.A, we construct a generalized Cantor dust $\mathcal{C}' \subset \mathbf{R}^2$ with $\Lambda^{2q}(\mathcal{C}')$ positive and finite. (Note that $\Lambda^p(\mathcal{C}') = \infty$ for any $p > 2q$.) According to Proposition 5.1, there is a finite distortion homeomorphism $h : \mathbf{R}^2 \to \mathbf{R}^2$ with $h(\mathcal{C}) = \mathcal{C}'$. Also, $\mu = q < 2C(\sigma)q$ (since $C(\sigma) = C(1/4) = \log 2$), so we are guaranteed that $\exp(\mu K_h)$ is locally integrable, where K_h is a distortion function for h.

Put $f = h \circ g$, a self-homeomorphism of $\mathbf{R}^2$ with $f(\mathbf{S}^1) \supset h(\mathcal{C}) = \mathcal{C}'$. According to Lemma 2.2 and its proof, f has finite distortion K_f, and since $\lambda = (1 - 2/p_0)\mu/K_0$, we also know that $\exp(\lambda K_f)$ is locally integrable (see the last paragraph of the proof of this lemma). Finally, if $p > k_2\lambda = 2q$, then $\Lambda^p(f\mathbf{S}^1) = \infty$ as asserted. □

5.C. Remark. Since we are restricting our attention to the plane $\mathbf{R}^2$, results of Astala [Ast94, Cor 1.2] permit us to take p_0 above arbitrarily close to $2K_0/(K_0-1)$. In principle one can calculate K_0 (e.g., by iterating a specific piecewise linear map) and thus obtain the estimate $k \leq 2K_0^2$ in connection with Problem A.

Added in proof. Problem C has recently been solved by Faraco, Koskela, and Zhong (Mappings of finite distortion: the degree of regularity, to appear). In particular the n-dimensional analog of Theorem A is true.

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