CONFORMAL DEFORMATIONS OF UNIFORM LOEWNER SPACES

DAVID A HERRON

Abstract. We study conformal deformations of uniform Loewner spaces, verify the Gehring-Hayman Inequality, demonstrate uniform continuity of the identity mapping, and provide results which describe the boundary of the deformed space. In addition, we establish two connections between Hausdorff content and modulus.

1. Introduction

Recently Bonk, Koskela and Rohde [BKR98] revealed how "to recover a remarkable part of geometric function theory" via their study of conformal metric densities on the unit ball in Euclidean space. With the hope of gaining further insight into the crucial properties of the underlying space, here we extend parts of their work to the setting of fairly general metric spaces. We utilize the ideas and concepts described in the recent monograph [BHK01] by Bonk, Heinonen and Koskela. The good news is that we can replace the Euclidean unit ball by any uniform Loewner metric measure space (Ω, d, μ) and retain surprisingly many of the [BKR98] results. There appear to be analogs of virtually all their results; we spotlight those given below primarily because either they are required for the proof of, or help to explain, Theorem E, which in turn is an essential ingredient for our later work. We point out that all these results are new even for the special case of uniform domains in Euclidean space. We do not prove analogs of [BKR98, Theorems 7.2,8.1].

Note that there are 'simple' examples of uniform Loewner spaces, which can even be taken to be topological balls in Euclidean space, for which the Bonk, Koskela, Rohde results fail to apply. One such construction begins with the so-called Rickman rug $S = J \times \mathbf{R}^{n-2} \subset \mathbf{R}^n$ where J is a blown-up middle-third von Koch snowflake arc; the complementary components of the $(n-1)$-surface S are uniform domains, but are not quasiconformal images of the unit ball (see [Tuk81, p.158]). A similar example, but with an especially nice boundary, is given by Semmes' construction in [Sem96, Theorem 1.12 & p.195].

Section 2 contains preliminary information including basic definitions and terminology descriptions, but we will mention a few items here. Following [BHK01], a *uniform space* is a locally compact, rectifiably connected and non-complete metric space (Ω, d) with the property that there is some constant $a \geq 1$ such that each pair of points can be joined by an a-uniform arc. A rectifiable curve γ joining x, y in Ω is an *a-uniform arc* provided

$$\ell(\gamma) \leq a|x-y|$$

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and

$$\min\{\ell(\gamma(x,z)), \ell(\gamma(y,z))\} \le a\, d(z) \qquad \text{for all } z \in \gamma.$$

Here $\ell(\gamma)$ is the arclength of γ; $|x-y| = d(x,y)$; $\gamma(x,z)$ denotes the subarc of γ between x, z; $d(x) = \operatorname{dist}(x, \partial\Omega)$; and $\partial\Omega = \bar\Omega \setminus \Omega$ where $\bar\Omega$ denotes the metric completion of Ω. These conditions on γ ensure that Ω contains the twisted double cone $\cup\{B(z;\lambda(z)/a) : z \in \gamma\}$ where $\lambda(z)$ is the left-hand-side of the above inequality. Uniform domains in Euclidean space were first studied by John [Joh61] and Martio and Sarvas [MS79] and their importance in function theory is well documented; see [Geh87], [Väi88]. More recently, uniform subdomains of the Heisenberg groups, as well as more general Carnot groups, have become a focus of study; see [GV95], [CT95], [CGN00], [Gre01].

Recall that the quasihyperbolic distance is defined by

$$k(x,y) = k_\Omega(x,y) = \inf \int_\gamma \frac{ds}{d(z)}$$

where the infimum is taken over all rectifiable curves which join x, y. We let $[x,y]$ denote a length-minimizing quasihyperbolic geodesic joining a pair of points x, y (this may not be unique). An important, and characteristic, property of uniform spaces is that quasihyperbolic geodesics are uniform arcs. (See [GO79, Theorems 1,2] for domains in Euclidean space and [BHK01, Theorem 2.10] for general metric spaces.) With this in mind, we agree that the phrase *a-uniform space* shall designate a uniform space in which the quasihyperbolic geodesics are a-uniform arcs. We mention that there is an alternate equivalent setting we could study. Namely, instead of asking that Ω be uniform, we could require that (Ω, k) be a Gromov hyperbolic space. In this situation we would appeal to the Bonk-Heinonen-Koskela theory [BHK01, Proposition 4.5] to obtain a uniform space 'equivalent' to our original Ω.

The primary object of our study here is the geometry of conformal deformations of our base space. Following [BKR98] and [BHK01], we consider a continuous positive 'density' ρ on Ω and we let $\Omega_\rho = (\Omega, d_\rho, \mu_\rho)$ denote the deformed space (see subsection 2.B). We call ρ a *conformal density* provided it satisfies both a uniform local Harnack type inequality,

$$\frac{1}{H} \le \frac{\rho(y)}{\rho(x)} \le H \qquad \text{for all } y \in B(x; d(x)/2), \tag{HI}$$

and a global regular volume growth estimate

$$\mu_\rho B_\rho(x;r) \le M\, r^Q \qquad \text{for all } r > 0. \tag{VG}$$

Both of these conditions are to hold for all points x in Ω. We use $H(\rho)$ or $M(\rho)$ to designate a Harnack constant for ρ or a mass constant for μ_ρ. The positive real number Q is generally the Hausdorff dimension of our space; it must agree with the number Q appearing in the definition of a Loewner space (a notion discussed in subsection 2.F).

Our first result is the so-called Gehring-Hayman Inequality (cf. [GH62]); it is an essential tool for most of what follows. This was proved in [BKR98, Theorem 3.1] for deformations of the Euclidean unit ball and in [HR93] for quasiconformal images of uniform domains in Euclidean space; see also [BBpp, Theorem 2.3], [BHK01, Chpt. 5] and [HN94].

Theorem A. *Let ρ be a conformal density on a uniform Loewner metric measure space (Ω, d, μ). Then there exists a constant Λ such that for all quasihyperbolic geodesics $[x,y]$ with endpoints in $\bar\Omega$,*

$$\ell_\rho[x,y] \le \Lambda\, d_\rho(x,y).$$

This result is quantitative: Λ depends only on the data associated with Ω.

Throughout this article the symbol Λ will stand for this Gehring-Hayman Inequality constant.

Here is a simple, but useful, consequence of the Gehring-Hayman Inequality: if there is an arc α joining some point w in Ω to some point ζ in $\partial\Omega$ with $\ell_\rho(\alpha) < \infty$, then $\ell_\rho(\gamma) < \infty$ for every quasihyperbolic geodesic ray going to ζ. In fact, as the following 'radial limit theorem' shows, this is actually true for 'mod_Q-a.e.' point of $\partial\Omega$. (The notion of Q-modulus zero is described at the end of subsection 2.F.)

Theorem B. *Let ρ be a conformal density on a uniform Loewner metric measure space (Ω, d, μ). Then there exists a Borel set $A \subset \partial\Omega$ with $\mathrm{mod}_Q A = 0$ and such that for any fixed $w \in \Omega$,*

$$\ell_\rho[w, \zeta) < \infty \qquad \textit{for all } \zeta \in \partial\Omega \setminus A.$$

A consequence of the above is that 'almost every' point of $\partial\Omega$ is d_ρ-rectifiably accessible, and we even have 'angular limits' at these points. See subsection 5.B and also subsection 6.A.

We next investigate properties of the identity map $id : \Omega_\rho \to \Omega$. Simple examples show that this may not be continuous if Ω is not bounded; see Example 5.5. Fortunately, the Bonk-Heinonen-Koskela theory permits us to replace any unbounded uniform space by a bounded one (first we quasihyperbolize, then we uniformize—see [BHK01, Theorem 3.6, Proposition 4.5]). Note that if Ω is a bounded a-uniform space, then $\bar\Omega$ is compact [BHK01, Proposition 2.20], so there exists a point $w \in \Omega$ with $d(w) = \max_{x\in\Omega} d(x)$, and $1 \le \operatorname{diam}\Omega / d(w) \le 2a$.

Theorem C. *Let ρ be a conformal density on a bounded uniform Loewner metric measure space (Ω, d, μ). Then there exists an increasing function ψ, with $\lim_{s\to 0}\psi(s) = 0$, such that for all continua E in Ω,*

$$\frac{\operatorname{diam} E}{\operatorname{diam}\Omega} \le \psi\left(\frac{\operatorname{diam}_\rho E}{\rho(w)d(w)}\right).$$

Here $w \in \Omega$ satisfies $d(w) = \max_{x\in\Omega} d(x)$. Moreover, when Ω is regular we can take $\psi(s) = C\, s^\alpha$ where $0 < \alpha \le 1$. This result is quantitative: ψ (and α, C) depend only on the data associated with Ω.

An obvious consequence of the above is that $id : \Omega_\rho \to \Omega$ is uniformly continuous (and even Hölder continuous when Ω is regular); see Corollary 5.6. This means that Cauchy sequences in Ω_ρ are also Cauchy in Ω, so we can identify $\bar\Omega_\rho$ with a subspace of $\bar\Omega$, and especially view $\partial_\rho\Omega = \bar\Omega_\rho \setminus \Omega_\rho$ as a subspace of $\partial\Omega$. Our next result permits us to describe $\partial_\rho\Omega$ precisely as the set of those points of $\partial\Omega$ which are d_ρ-rectifiably accessible; see Corollary 6.1

Theorem D. *Let ρ be a conformal density on a bounded uniform locally Loewner metric measure space (Ω, d, μ). Suppose that $\{x_k\}$ is a Cauchy sequence in Ω_ρ which converges in $\bar\Omega$ to some point $\xi \in \partial\Omega$. Then*

$$\ell_\rho[w, \xi) < \infty.$$

Moreover, if $\{y_k\}$ is any sequence along $[w, \xi)$ which converges to ξ, then $d_\rho(x_k, y_k) \to 0$. Here w is any fixed point in Ω.

We next turn our attention to estimating $d_\rho(x) = \operatorname{dist}_\rho(x, \partial_\rho\Omega)$ in terms of $\rho(x)d(x)$. A lower bound is immediate via the Harnack inequality. To obtain an upper bound, we at least need $\partial_\rho\Omega \ne \emptyset$. According to Theorems B and G this will be the case if $\dim_{\mathcal H}(\partial\Omega) > 0$. However, even then we may have $d_\rho(x) \to \infty$ while $\rho(x)d(x)$ stays bounded; see the discussion in subsection 6.A. We require a condition which ensures that Ω has a uniformly thick boundary as seen from each point. With this in mind, we introduce the following notion: we say that (Ω, d, μ) satisfies a *Whitney ball modulus property* if there exists a constant $m > 0$ such that

$$\operatorname{mod}_Q(\bar{B}(x; d(x)/2), \partial\Omega; \Omega) \geq m \qquad \text{for all } x \in \Omega.$$

Theorem E. *Let ρ be a conformal density on a uniform Loewner metric measure space (Ω, d, μ). Suppose that Ω enjoys a Whitney ball modulus property. Then there is a constant b such that for all $x \in \Omega$,*

$$\frac{1}{2H}\rho(x)d(x) \leq d_\rho(x) \leq bH\rho(x)d(x);$$

the constant b depends only on the data associated with Ω.

We mention that any uniform Loewner space with connected boundary satisfies a Whitney ball modulus property, provided it and its boundary are simultaneously bounded or unbounded. Similarly any bounded uniform Loewner space with a finite number of non-degenerate boundary components will enjoy this modulus property. See subsection 6.A. However, here is a sufficient condition for this property to hold which allows for a totally disconnected boundary.

Theorem F. *Let (Ω, d, μ) be a locally Loewner, uniform metric measure space. Assume Ω and $\partial\Omega$ are either both bounded or both unbounded. Suppose that for some $p > 0$, $\partial\Omega$ satisfies the Hausdorff p-content condition*

$$\mathcal{H}^p(\partial\Omega \cap \bar{B}(\zeta; r)) \geq c\, r^p \qquad \text{for all } 0 < r \leq \operatorname{diam}(\partial\Omega) \text{ and all } \zeta \in \partial\Omega.$$

Then Ω enjoys a Whitney ball modulus property with a constant m which depends only on c and the data associated with Ω.

In contrast to the Euclidean case, the converse to the above is false. Example 3.2 furnishes a space with an isolated boundary point which nonetheless satisfies a Whitney ball modulus property. This is also a strong counter-example for the converse of the following related result.

Theorem G. *Suppose a compact set A in a uniform locally Loewner metric measure space has zero modulus. Then the Hausdorff dimension of A is zero.*

An easy consequence of Theorem E above is that the quasihyperbolizations of Ω and Ω_ρ are bilipschitz equivalent, and it follows that (Ω_ρ, k_ρ) is a Gromov hyperbolic space; see subsection 6.C. The subdomains of Euclidean space which are Gromov hyperbolic with respect to their quasihyperbolic metrics can be characterized by two properties: the Gehring-Hayman Inequality and a Separation Property (see [BBpp], [BHK01, p.75]). We close this Introduction with the following result which says that the conformal deformation of a uniform Loewner space also satisfies the aforementioned Separation Property. This property asserts that $\partial_\rho\Omega$ cannot contain outward pointing 'ridges', although outward pointing 'spires' are permitted.

Theorem H. *Let ρ be a conformal density on a uniform Loewner metric measure space (Ω, d, μ). Suppose that Ω enjoys a Whitney ball modulus property. There is a constant C such that for all $\zeta \in \partial\Omega$ and all $x \in [w, \zeta)$,*

$$A \cap B_\rho(x, Cd_\rho(x)) \neq \emptyset$$

for any arc A in Ω which joins $[w, x]$, $[x, \zeta)$. Here w is any fixed point in Ω, and, as always, the result is quantitative: C depends only on the data associated with Ω.

This document is organized as follows: Section 2 contains preliminary information including basic definitions and terminology descriptions as well as elementary and/or known facts. In particular, subsection 2.F contains useful properties of Loewner spaces including the concept of zero modulus (which is discussed at the end). In Section 3 we study the Whitney ball modulus property and prove Theorems F and G. We verify the Gehring-Hayman Inequality (Theorem A) in Section 4. Section 5 is devoted to explaining and corroborating Theorems B, C and D. We establish Theorems E and H in Section 6. We conclude with Section 7 where we define regular spaces and exhibit pointwise estimates for a conformal density.

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2. Preliminaries

2.A. **General Information.** Our notation is relatively standard and conforms with that of [BHK01]. We work in the setting of a metric measure space (Ω, d, μ) with Ω a locally compact, rectifiably connected, non-complete metric space and μ a Borel regular measure with dense support. We let $|x - y| = d(x, y)$ denote the distance between points x, y and $B(x; r) = \{y : |x - y| < r\}$ ($S(x; r) = \{y : |x - y| = r\}$) is the open ball (sphere) of radius r centered at the point x. When $B = B(x; r)$, $cB = B(x; cr)$. We write $c = c(a, \dots)$ to indicate a constant c which depends only on the parameters $a, \dots$, and $a \approx b$ means there exists a positive finite constant c with $1/c \le a/b \le c$. Typically a, b, c, C, K will be constants which depend on various parameters, and we try to make this as clear as possible often giving explicit values.

2.B. **Conformal Deformations.** Given a positive continuous 'density' ρ on Ω we let

$$\ell_\rho(\gamma) = \int_\gamma \rho \, ds$$

denote the ρ-length of a rectifiable curve γ. Then the ρ-distance between two points x, y is defined by

$$d_\rho(x, y) = \inf\{\ell_\rho(\gamma) : \gamma \text{ a rectifiable curve joining } x, y \text{ in } \Omega\}.$$

The ρ-balls (etc.) are written as $B_\rho(x; r)$; these are the metric balls in Ω_ρ, so $B_\rho(x; r) = \{y \in \Omega : d_\rho(x, y) < r\}$. We also form a new measure μ_ρ given by

$$\mu_\rho(E) = \int_E \rho^Q \, d\mu.$$

Here Q is generally the Hausdorff dimension of Ω (see Section 3); it must agree with the number Q appearing in the definition of a Loewner space. Thus each density ρ gives rise to a new metric measure space $\Omega_\rho = (\Omega, d_\rho, \mu_\rho)$ which we call a conformal deformation of Ω. We already mentioned one such density in the Introduction; the quasihyperbolic distance is determined via the density $1/d$. Note however that in general $1/d$ is not a conformal density because it does not satisfy the volume growth requirement (VG). We refer to [BKR98, Section 2] for many examples of conformal densities; we mention that $\rho = |f'|$ is a conformal density for any conformal map f defined in a subdomain Ω of the complex plane.

2.C. **Quasihyperbolic Distance and Geodesics.** We remind the reader of the following basic estimates for quasihyperbolic distance, first established by Gehring and Palka [GP76, Lemma 2.1]:

$$k(x,y) \geq j(x,y) = \log\left(1 + \frac{|x-y|}{\min\{d(x),d(y)\}}\right) \geq \left|\log\frac{d(x)}{d(y)}\right|. \tag{2.1}$$

See also [BHK01, (2.3),(2.4)]. Another crucial piece of information is a characterization of uniformity due to Gehring and Osgood [GO79, Theorems 1,2]; Bonk, Heinonen, and Koskela [BHK01, Lemma 2.13] verified the necessity of this condition for the metric space setting, while the Gehring-Osgood argument can be modified to establish the sufficiency.

2.2. Fact. *A locally compact, rectifiable, non-complete metric space is a-uniform if and only if $k(x,y) \leq b\, j(x,y)$ for all points x, y. The constants a and b depend only on each other. In fact, one can take $b = 4a^2$.*

Recall that a *quasihyperbolic geodesic* is the image $f(I)$ of some isometric embedding $f : I \to (\Omega, k)$ where $I \subset \mathbf{R}$ is an interval; we use the adjectives *segment, ray,* or *line* (respectively) to indicate that I is bounded, semi-infinite, or all of $\mathbf{R}$. We use the notation $[x,y]$ to mean a (not necessarily unique) geodesic segment (or ray or line) joining points $x, y \in \Omega$ (or $\bar{\Omega}$).

We need to know that boundary points in a uniform space can be joined by quasihyperbolic geodesics, and that these geodesics are still uniform arcs. This is a consequence of the fact that a uniform space Ω has a proper, geodesic, and Gromov hyperbolic quasihyperbolization (Ω, k) (see [BHK01, p.10, Proposition 2.20, Theorem 3.6]). A straightforward application of the Arzela-Ascoli theorem (cf. [Mun00, 47.1, p.290 & 46.2, p.283]) establishes the existence of quasihyperbolic geodesic rays joining an interior point to a boundary point (e.g., see [BHK01, Proposition 3.12]), however, the construction for joining two boundary points with a quasihyperbolic geodesic line is a tad more delicate. It is not hard to modify and apply the proof of [BH99, Lemma 3.2, p.428]; alternatively, we can simply use this result in conjunction with the fact that uniform spaces are Gromov hyperbolic. It is then easy to verify that these limit geodesics are uniform arcs.

2.D. **Harnack Inequalities.** The Harnack inequality (HI), as stated, only requires that ρ be essentially constant on Whitney type balls. We can do the usual chaining type arguments to see that such a density ρ will satisfy a Harnack type inequality on much bigger sets, of course with a change in the Harnack constant. Here is a useful example of this phenomena.

2.3. Fact. *Let ρ be a continuous positive density on an a-uniform metric space (Ω, d). Assume that ρ satisfies* (HI). *Then for all $x, y \in \Omega$ with $k(x,y) \leq K$, $1/H_1 \leq \rho(y)/\rho(x) \leq H_1$, where $H_1 = H^{1+3K}$.*

Here is a simple result which illustrates the usefulness of requiring that ρ satisfy (HI).

2.4. Lemma. *Let ρ be a continuous positive density, which satisfies* (HI)*, on an a-uniform metric space (Ω, d). Then for all $z \in \Omega$,*

$$\frac{1}{H} \leq \frac{\ell_\rho[x,y]}{\rho(z)|x-y|} \leq aH \quad \textit{for all } x, y \in B(z; d(z)/4a),$$

where $H = H(\rho)$.

Proof. Fix $x, y \in B(z; d(z)/4a)$. Let $u \in [x, y]$. Then $\min\{|x-u|, |y-u|\} \le a|x-y|/2 \le d(z)/4$, so

$$|z-u| \le \min\{|z-x| + |x-u|, |z-y| + |y-u|\} \le (d(z)/4a) + d(z)/4 \le d(z)/2.$$

Thus $[x, y] \subset B(z; d(z)/2)$ and so by (HI) we have

$$\ell_\rho[x, y] \le H\rho(z)\ell[x, y] \le aH\rho(z)|x-y|$$

and similarly

$$\ell_\rho[x, y] \ge \frac{1}{H}\rho(z)\ell[x, y] \ge \frac{1}{H}\rho(z)|x-y|.$$

$\diamond$

2.5. Corollary. *Let ρ be a continuous positive density, which satisfies* (HI)*, on an a-uniform metric space (Ω, d). Then for all $z \in \Omega$,*

$$d_\rho(x, y) \le aH\rho(z)|x-y| \quad \textit{for all } x, y \in B(z; d(z)/4a),$$

where $H = H(\rho)$. In particular, $\operatorname{diam}_\rho B(z; d(z)/4a) \le \frac{H}{2}\rho(z)d(z)$.

2.E. **Modulus.** For $p \ge 1$ we define the *p-modulus* of a family Γ of curves in Ω by

$$\operatorname{mod}_p \Gamma = \inf \int \rho^p \, d\mu,$$

where the infimum is taken over all $\rho \in \operatorname{adm}(\Gamma)$, the class of all *admissible* densities for Γ; i.e. all non-negative Borel functions $\rho : \Omega \to [0, \infty]$ satisfying $\int_\gamma \rho \, ds \ge 1$ for all locally rectifiable curves $\gamma \in \Gamma$. Then the *p-modulus* of a pair of disjoint, compact sets $E, F \subset \bar{D}$, relative to an open, connected subset $D \subset \Omega$, is

$$\operatorname{mod}_p(E, F; D) = \operatorname{mod}_p \Delta(E, F; D)$$

where $\Delta(E, F; D)$ is the family of all curves joining the sets E, F in D. (Here $\bar{D}$ denotes the closure of D relative to $\bar{\Omega}$, and a curve in D joins E, F provided its closure, relative to $\bar{\Omega}$, meets both E, F; see [Väi71, pp.21,23].) We also let $\operatorname{adm}(E, F; D) = \operatorname{adm} \Delta(E, F; D)$.

For the reader's convenience, we cite the following standard modulus estimates. First we have the so-called Comparison Principle; see [Vuo88, 5.35, p.61].

2.6. Fact. *Let $E, F, A, B \subset D$ be compact subsets of a subdomain D of Ω. Then*

$$\operatorname{mod}_p(E, F; D) \ge 3^{-p} \min\{\operatorname{mod}_p(E, A; D), \operatorname{mod}_p(F, B; D), I\},$$

where

$$I = \inf\{\operatorname{mod}_p(\alpha, \beta; D) : \alpha \in \Delta(E, A; D), \beta \in \Delta(F, B; D)\}.$$

Next we record the Spherical Ring Estimate; see [HK98, 3.14,p.17].

2.7. Fact. *Let $x \in \Omega$, $0 < 2r \le R$, and suppose that the upper mass condition $\mu B(x; t) \le Mt^p$ holds for all $0 < t < r + R$. Then*

$$\operatorname{mod}_p(\bar{B}(x; r), \Omega \setminus B(x; R); \Omega) \le C \left(\log \frac{R}{r}\right)^{1-p},$$

where $C = 2^{p+1}M/\log 2$.

Finally, we require the following Basic Modulus Estimate; see [BKR98, Lemma 3.2].

2.8. Fact. *Let (Ω, d, μ) be a metric measure space. Assume that ρ is a continuous positive density on Ω which satisfies the volume growth condition* (VG) *at some point $x \in E \subset \Omega$. Suppose that $L > \lambda \geq \operatorname{diam}_\rho E$, and that Γ is some family of curves γ in Ω each having one endpoint in E and satisfying $\ell_\rho(\gamma) \geq L$. Then*

$$\operatorname{mod}_Q \Gamma \leq C\left(\log\left(1+\frac{L}{\lambda}\right)\right)^{1-Q},$$

where $C = 2^{Q+1}M(\rho)/\log 2$.

2.9. Corollary. *Let (Ω, d, μ) be a metric measure space. Assume that for some point $x \in E \subset \Omega$, the upper mass condition $\mu B(x;r) \leq Mr^Q$ holds for all $r > 0$. Suppose that Γ is a family of curves γ in Ω each having one endpoint in E and satisfying $\ell(\gamma) \geq L > \operatorname{diam} E$. Then*

$$\operatorname{mod}_Q \Gamma \leq C\left(\log\left(1+\frac{L}{\operatorname{diam} E}\right)\right)^{1-Q},$$

where $C = 2^{Q+1}M/\log 2$.

2.F. **Loewner Spaces.** The notion of a Loewner space was introduced by Heinonen and Koskela in their study [HK98] of quasiconformal mappings of metric spaces; Heinonen's recent monograph [Hei01] renders an enlightening account of these ideas. A path-connected metric measure space (Ω, d, μ) is a *Q-Loewner space* provided there is some positive function $\varphi(t)$ with the property that for all non-degenerate disjoint continua E, F in Ω,

$$\operatorname{mod}_Q(E, F; \Omega) \geq \varphi(t) \qquad \text{whenever } \Sigma(E, F) \leq t,$$

where

$$\Sigma(E, F) = \operatorname{dist}(E, F)/\min\{\operatorname{diam}(E), \operatorname{diam}(F)\}$$

is the relative size of the pair E, F. In principle, Q could be any positive real number, but in fact we always have $Q \geq \dim_{\mathcal{H}}(\Omega) \geq 1$.

The natural optimal *Loewner control function* is

$$\varphi(t) := \inf\{\operatorname{mod}_Q(E, F; \Omega) : \Sigma(E, F) \leq t\},$$

where E, F are non-degenerate disjoint continua in Ω; this choice gives us a control function $\varphi(t)$ which decreases as t increases. (Of course our space is Loewner precisely when the above function is positive for all $t > 0$.) We mention that in the above definition it actually suffices to have the seemingly weaker requirement that

$$\operatorname{mod}_Q(E, F; \Omega) \geq \varphi(t) \qquad \text{whenever } \Sigma(E, F) = t;$$

one sees this by examining certain subcontinua, but some care is required! (The reader interested in verifying this statement may find [Kur68, Theorem 1, p.172] or [HY88, Theorem 2-16, p.47] useful.)

Suppose we know that our Loewner control function satisfies $\lim_{t\to\infty} \varphi(t) = 0$. In this setting we can define

$$\varphi^{-1}(s) := \inf \varphi^{-1}(0, s],$$

and we see that $\lim_{s\to 0} \varphi^{-1}(s) = \infty$. This condition on φ certainly need not always be true, but the Spherical Ring Estimate can be used to see that it does hold if there is some point x such that the upper mass condition $\mu B(x;r) \leq Mr^Q$ holds for all sufficiently small radii. More generally, we have the following information which will be used later.

2.10. Lemma. *Let (Ω, d, μ) be a locally compact non-complete Q-Loewner metric measure space. Suppose that there exists a continuous density ρ on Ω which is 'conformal at' some point $x \in \Omega$ (meaning that (HI), (VG) hold at x). Assume further that Ω is locally quasi-convex at x in the sense that there exist constants $a \geq 1$, $s > 0$ such that any pair of points $y, z \in B(x; s)$ can be joined by an arc α with $\ell(\alpha) \leq a\,|y-z|$. Then the Loewner control function satisfies $\lim_{t\to\infty} \varphi(t) = 0$.*

Proof. Put $d = d(x) > 0$ and choose a point $w \in \Omega$ with $|x-w| \geq 3d/4$. Let C be a curve which joins x to w. Let F be the w-component of $C \setminus B(x; d/2)$. Consider $0 < r \leq s$ much less than $d/4$ and let E_r be the x-component of $C \cap \bar{B}(x; r)$. Fix points $y, z \in E_r$ and let α be the promised arc joining them with $\ell(\alpha) \leq a|y-z|$. Then $\alpha \subset B(x; d/2)$ (provided r is small enough), so

$$d_\rho(y,z) \leq \ell_\rho(\alpha) \leq H\rho(x)\ell(\alpha) \leq 2aHr\,\rho(x),$$

and therefore $\operatorname{diam}_\rho(E_r) \leq \lambda = 2aHr\,\rho(x)$.

On the other hand, any curve $\gamma \in \Delta(E_r, F; \Omega)$ contains a subarc β joining $S(x;r)$ to $S(x; d/2)$ in $B(x; d/2)$, so

$$\ell_\rho(\gamma) \geq \ell_\rho(\beta) \geq \frac{\rho(x)}{H}\left(\frac{d}{2} - r\right) = L.$$

Since $L > \lambda$ (provided r is small enough), the Basic Modulus Estimate yields

$$\operatorname{mod}_Q(E_r, F; \Omega) \leq c\,(\log(1 + L/\lambda))^{1-Q}.$$

We also have $\operatorname{diam}(F) \geq d/4 \geq 2r \geq \operatorname{diam}(E_r) \geq r$, so $\Sigma(E_r, F) \leq d/2r = t$ and thus according to the Loewner criteria,

$$\varphi(t) \leq \operatorname{mod}_Q(E_r, F; \Omega) \leq c\left(\log\left(1 + \frac{t-1}{2aH^2}\right)\right)^{1-Q}.$$

Clearly the right-hand-side above tends to zero as $r \to 0$ (since $t = d/2r \to \infty$). ◇

We use the terminology *locally Q-Loewner* to mean that there exists $\kappa \geq 1$, $\varepsilon_0 \in (0, 1/\kappa]$ and $m > 0$ with the property that for all $0 < \varepsilon \leq \varepsilon_0$, all $x \in \Omega$ and all non-degenerate disjoint continua E, F in $B(x; \varepsilon\, d(x))$,

$$\Sigma(E,F) \leq 16 \implies \operatorname{mod}_Q(E, F; B(x; \kappa\varepsilon\, d(x))) \geq m.$$

In our setting, where Ω is a uniform space, this local Loewner condition actually implies a global one; this non-trivial result is due to Bonk, Heinonen and Koskela [BHK01, Theorem 6.4, Remark 6.38, Proposition 6.49]. To be more precise, we record this information below. We utilize the ideas behind the proof of [BHK01, Theorem 6.4] in our proofs of Theorems F, G. (Actually, the germ of these ideas can be found already in Gehring-Martio's proof that quasiextremal distance domains are quasiconvex [GM85, Theorem 2.7].) It is easy to see that a Loewner space is locally Loewner provided our measure locally satisfies an upper mass condition (so that we can use the Spherical Ring Estimate in Fact 2.7); however, this seems to be a delicate issue when no such mass condition holds.

2.11. Fact. *Let (Ω, d, μ) be an a-uniform locally Q-Loewner (with data $\kappa \geq 1$, $\varepsilon_0 \in (0, 1/\kappa]$, $m > 0$, $Q > 0$) metric measure space. Then:*

(a) *there exists a function* ψ *depending only on the data associated with* Ω *such that*

$$\operatorname{mod}_Q(E, F; B(x; \kappa'\varepsilon' d(x))) \geq \psi(t) \qquad \text{whenever } \Sigma(E, F) \leq t$$

for non-degenerate disjoint continua E, F *in* $B(x; \varepsilon' d(x))$, *where* $\kappa' = 1 + 2a + 6\kappa$ *and* $0 < \varepsilon' \leq \varepsilon'_0 = \min\{\varepsilon_0, (2 + 4a + 6\kappa)^{-1}\}$.

(b) Ω *is* Q*-Loewner with a control function* $\varphi(t)$ *which depends only on the original data associated with* Ω.

(c) *we even have* $\operatorname{mod}_Q(E, F; \Omega) \geq \varphi(t)$ *when* $\Sigma(E, F) \leq t$ *for all non-degenerate disjoint continua* E, F *in* $\bar\Omega$.

There is one more result we require which also follows from the proof of [BHK01, Theorem 6.4].

2.12. Fact. *Let* (Ω, d, μ) *be an* a*-uniform locally* Q*-Loewner metric measure space. Suppose* E, F *are non-degenerate connected subsets of* $\bar\Omega$ *with* $\bar E \cap \bar F \neq \emptyset$. *Then* $\operatorname{mod}_Q(E, F; \Omega) = \infty$.

I thank Juha Heinonen for outlining the argument here. Choose a point $z \in \bar E \cap \bar F$ and fix $0 < r < (1/2)\min\{\operatorname{diam} E, \operatorname{diam} F\}$. Let E', F' be subcontinua of E, F respectively which join the spheres $S(z; r)$, $S(z; r/2)$ inside the ring $\bar B(z; r) \setminus B(z; r/2)$ (see [HY88, Theorem 2-16] or [Kur68, Theorem 1, p.172] for the existence of such continua). We claim that there are constants $b \geq 2$ and $\delta > 0$ such that

$$\operatorname{mod}_Q(E', F'; B(z; br) \setminus B(z; r/b)) \geq \delta.$$

Then by choosing $r_1 > r_2 > \cdots > r_i \to 0$ so that the spherical rings $B(z; br_i) \setminus B(z; r_i/b)$ are all disjoint, we see that

$$\operatorname{mod}_Q(E, F; \Omega) \geq \sum_i \operatorname{mod}_Q(E'_i, F'_i; B(z; br_i) \setminus B(z; r_i/b)) = \infty$$

as desired. The Loewner condition gives $\operatorname{mod}_Q(E', F'; \Omega) \geq \varphi(2)$, since $\operatorname{dist}(E', F') \leq r \leq 2\min\{\operatorname{diam} E', \operatorname{diam} F'\}$. However, a careful look at the proof of [BHK01, Theorem 6.4] reveals that they actually establish $\operatorname{mod}_Q(E', F'; B(z; br) \setminus B(z; r/b)) \geq \delta$ where $b = b(a)$ and δ depends only on the data associated with Ω.

We conclude this section with the following definition: we say that a compact set $A \subset \bar\Omega$ has Q-modulus *zero*, denoted by $\operatorname{mod}_Q A = 0$, provided $\operatorname{mod}_Q(E, A; \Omega) = 0$ for some non-degenerate continuum $E \subset \Omega \setminus A$. Here we are tacitly assuming that our space is Q-Loewner, in particular so that $\operatorname{mod}_Q(E, A; \Omega)$ is simultaneously either zero or non-zero for all non-degenerate continua $E \subset \Omega \setminus A$. It is not difficult to verify this, so let us check the details. (We thank the referee for pointing out a simpler proof!)

2.13. Lemma. *Let* $A \subset \bar\Omega$ *be compact, where* Ω *is a* Q*-Loewner metric measure space (and* $\bar\Omega = \Omega$ *if* Ω *is already complete). If* $\operatorname{mod}_Q(E, A; \Omega) = 0$ *for some non-degenerate continuum* $E \subset \Omega \setminus A$, *then* $\operatorname{mod}_Q(F, A; \Omega) = 0$ *for every non-degenerate continuum* $F \subset \Omega \setminus A$.

Proof. We make repeated use of the following fact (see [Fug57, Theorem 2, p.179]): $\operatorname{mod}_p \Gamma = 0$ if and only if there exists $\rho \in L^p(\Omega)$ with $\int_\gamma \rho\, ds = \infty$ for all locally rectifiable $\gamma \in \Gamma$. Let $E \subset \Omega \setminus A$ be a non-degenerate continuum and assume that $\operatorname{mod}_Q(E, A; \Omega) = 0$. Select $\rho \in L^Q(\Omega)$ with $\int_\gamma \rho\, ds = \infty$ for all locally rectifiable $\gamma \in \operatorname{adm}(A, E; \Omega)$.

Suppose $\operatorname{mod}_Q(F, A; \Omega) > 0$ for some non-degenerate continuum $F \subset \Omega \setminus A$. This time the above fact guarantees the existence of a locally rectifiable $\alpha \in \operatorname{adm}(A, F; \Omega)$ with $\int_\alpha \rho\, ds < \infty$. It must be that $\alpha \cap E = \emptyset$, and so the Loewner property asserts that $\operatorname{mod}_Q(\alpha, E; \Omega) > 0$.

Employing the above fact once more we obtain a locally rectifiable $\beta \in \operatorname{adm}(\alpha, E; \Omega)$ with $\int_\beta \rho\, ds < \infty$. Now $\alpha \cup \beta$ contains a locally rectifiable $\gamma \in \operatorname{adm}(A, E; \Omega)$ with $\int_\gamma \rho\, ds < \infty$, which contradicts our original assumption. ◇

Notice that Example 3.2 exhibits a space with an isolated boundary point which does not have modulus zero.

3. Hausdorff Content and Modulus

In this section we establish Theorems F and G which relate Hausdorff content and modulus. These facts are well-known for subsets of Euclidean space [Wal77] and can be established in a similar way for subsets of a doubling metric space (cf. [HK98, Theorem 5.9]). The classical proofs go by way of Frostman's lemma. However, in our setting this tool is unavailable; note that we make no assumptions about our underlying measure—in particular no doubling nor upper mass conditions are assumed. We also analyze the Whitney ball modulus property and present an interesting example.

3.A. **Preliminaries.** Recall that *Hausdorff p-dimensional content* is given by

$$\mathcal{H}^p(A) = \inf \sum \operatorname{diam}(E_i)^p$$

where the infimum is taken over all countable families $\{E_i\}$ with $A \subset \cup E_i$. Then the *Hausdorff dimension* of A is determined by

$$\dim_{\mathcal{H}}(A) = \inf\{p > 0 : \mathcal{H}^p(A) = 0\}.$$

Our arguments for Theorems F and G rely heavily on the Bonk-Heinonen-Koskela method of proof that uniform locally Loewner spaces are globally Loewner [BHK01, Theorem 6.4]. (The diligent reader will have already perused this argument to check Facts 2.11 and 2.12.) Notice that this technique serves as a replacement for Frostman's lemma which is not available in our setting. The following information is required; see [BHK01, Lemma 6.9, Remark 6.38].

3.1. Fact. *Let $[x, z]$ be a quasihyperbolic geodesic in an a-uniform space Ω with $x \in \Omega$, $z \in \bar{\Omega}$ and $k(x, z) \geq \varepsilon$ for some $0 < \varepsilon \leq 1/8$. Then there are successive points $z_0 = x, z_1, z_2, \ldots,$ along $[x, z]$, with $z = z_N$ for some N if $z \in \Omega$ and $z_n \to z$ if $z \in \partial\Omega$, and such that*

$$\varepsilon \leq k(z_{n-1}, z_n) \leq 2\varepsilon \quad \text{for all } n \geq 1,$$

and

$$\sum_n d(z_n)^\alpha \leq C(\alpha, \varepsilon, a)\, (\ell[x, z])^\alpha \quad \text{for all } \alpha > 0.$$

When $z \in \partial\Omega$ we get $k(z_{n-1}, z_n) = \varepsilon$ for all $n \geq 1$.

3.B. **Whitney Ball Modulus Property.** Recall that Ω has the Whitney ball modulus property, abbreviated WBMP, provided there exists a constant $m > 0$ such that for all $x \in \Omega$,

$$\operatorname{mod}_Q(\bar{B}(x; d(x)/2), \partial\Omega; \Omega) \geq m.$$

It is not hard to see that any uniform (locally) Loewner space with connected boundary satisfies this condition with a constant $m = \min\{\varphi(2), \varphi(\operatorname{diam}\Omega / \operatorname{diam}\partial\Omega)\}$, provided the space and its boundary are either both bounded or both unbounded. (The domain $\Omega = \mathbf{R}^n \setminus \bar{\mathbf{B}}^n$ has $\partial\Omega = \mathbf{S}^{n-1}$ which is connected, but Ω fails to have WBMP.) Similarly, if Ω

is bounded and $\partial\Omega$ has finitely many non-degenerate connected components C_i say, then Ω has the WBMP with a constant $m = \min\{\varphi(4), \varphi(\operatorname{diam}\Omega/\min\operatorname{diam}C_i)\}$.

It is natural to ask what happens if Ω has an isolated boundary point. If we have 'spherical ring estimates' for rings centered at such a point (e.g., if we have an upper mass condition for balls centered at this point), then it is easy to see that the WBMP fails to hold. On the other hand, the following example illustrates that it is possible for a space to possess the Whitney ball modulus property and have isolated boundary points. This example also shows, among other things, that the WBMP is a true metric measure space condition.

3.2. Example. *Let Ω be the punctured unit disk in $\mathbf{R}^2$, $\Omega = \{0 < |z| < 1\}$, with Euclidean distance, but with measure μ given by $d\mu(z) = dx\,dy/|z| = dr\,d\theta$ (here $z = x + iy = re^{i\theta}$). Then Ω enjoys the Whitney ball modulus property. Also, $\{0\}$ does* not *have 2-modulus zero.*

Proof. Since we are using Euclidean distance, the admissible densities for determining the modulus in Ω are just the 'usual' Euclidean ones. Also, since $\iint \rho^2\,d\mu \geq \iint \rho^2\,dxdy$, we see that the 2-modulus in Ω is at least as large as the Euclidean 2-modulus in the punctured unit disk. In particular we deduce that Ω is 2-Loewner. Similarly, the WBMP condition is easily seen to hold for points in $1/2 \leq |z| < 1$.

Consider a point z with $0 < |z| < 1/2$. Let $A = \{0\}$, $B = \bar{B}(z; |z|/2)$ and $C = S(0; |z|/4)$. The Comparison Principle (Fact 2.6) gives us

$$\operatorname{mod}_2(B, A; \Omega) \geq \frac{1}{9}\min\{\operatorname{mod}_2(B, C; \Omega), \operatorname{mod}_2(A, C; \Omega), I\},$$

where

$$I = \inf\{\operatorname{mod}_2(\alpha, \beta; \Omega) : \alpha \in \Delta(B, C; \Omega), \beta \in \Delta(A, C; D)\}.$$

The Loewner condition ensures that $\operatorname{mod}_2(B, C; \Omega) \geq \varphi(1/2)$ and that $I \geq \varphi(2)$.

It remains to examine $\operatorname{mod}_2(A, C; \Omega)$. Let $\rho \in \operatorname{adm}(A, C; \Omega)$. For each $0 < \theta < 2\pi$, the radius $\gamma_\theta = (0, te^{i\theta})$ belongs to $\Delta(A, C; \Omega)$, where $t = |z|/4$. The Cauchy-Schwarz inequality now gives us

$$1 \leq \left(\int_{\gamma_\theta} \rho\,ds\right)^2 \leq \int_0^t \rho(re^{i\theta})^2\,dr \cdot \int_0^t dr \leq \frac{1}{8}\int_0^t \rho(re^{i\theta})^2\,dr,$$

so

$$\iint_\Omega \rho^2\,d\mu \geq \int_0^{2\pi}\int_0^t \rho(re^{i\theta})^2\,dr\,d\theta \geq 16\pi.$$

$\diamond$

3.C. **Proof of Theorem F.** Let (Ω, d, μ) be a locally Q-Loewner, a-uniform metric measure space. Suppose that Ω and $\partial\Omega$ are either both bounded or both unbounded and that $\partial\Omega$ has positive (Hausdorff) p-density for some $p > 0$. We verify the Whitney ball modulus property for Ω. According to Fact 2.11, there exist $\kappa \geq 6$, $0 < \varepsilon_0 \leq 1/\kappa$, and ψ such that for all $x \in \Omega$ and all $t > 0$,

$$\operatorname{mod}_Q(E, F; B(x; \kappa\varepsilon_0\,d(x))) \geq \psi(t) \qquad \text{whenever } \Sigma(E, F) \leq t$$

for all non-degenerate disjoint continua E, F in $B(x; \varepsilon_0\,d(x))$.

Let $x \in \Omega$, pick $\xi \in \partial\Omega$ with $d = d(x) = |x - \xi|$, and put $B = \bar{B}(x; d/2)$, $A = \partial\Omega \cap \bar{B}(\xi; d)$. We demonstrate that $\operatorname{mod}_Q(B, A; \Omega) \geq m$ where $m > 0$ depends only on the data associated with Ω.

We begin by explaining why we may assume that $d \leq \operatorname{diam}(\partial\Omega)/2a$. Assume that both Ω, $\partial\Omega$ are bounded, and that we have verified the desired modulus inequality for balls $\bar{B}(y; d(y)/2)$ when $d(y) \leq b\,\mathrm{diam}(\partial\Omega)$, where $b > 0$ is some constant. Assume $d > b\,\mathrm{diam}(\partial\Omega)$. Choose a point $x' \in [x, \xi)$ with $d' = d(x') = b\,\mathrm{diam}(\partial\Omega)/5$ and let $B' = \bar{B}(x'; d'/2)$. The Comparison Principle (Fact 2.6) now gives us

$$\mathrm{mod}_Q(B, \partial\Omega; \Omega) \geq 3^{-Q} \min\{\mathrm{mod}_Q(B, B'; \Omega), \mathrm{mod}_Q(B', \partial\Omega; \Omega), I\},$$

where $I = \inf\{\mathrm{mod}_Q(\alpha, \beta; D) : \alpha \in \Delta(B, B'; \Omega), \beta \in \Delta(B', \partial\Omega; \Omega)\}$. By assumption $\mathrm{mod}_Q(B', \partial\Omega; \Omega) \geq m$. As $\mathrm{diam}(B) \geq \mathrm{diam}(B') \geq d'/2$, we find that B and B' have relative size $\Sigma(B, B') \leq 10a\,\mathrm{diam}(\Omega)/[b\,\mathrm{diam}(\partial\Omega)]$, and therefore

$$\mathrm{mod}_Q(B, B'; \Omega) \geq \varphi(10a\,\mathrm{diam}(\Omega)/[b\,\mathrm{diam}(\partial\Omega)]).$$

(By considering the example where Ω is a sphere with a spherical cap removed, we see that we must have $m \to 0$ as $\mathrm{diam}(\partial\Omega)/\mathrm{diam}(\Omega) \to 0$, so an estimate as above is not surprising.) Finally, to estimate I we use the observation that $\Sigma(\alpha, \beta) \leq 2$ for any curves $\alpha \in \Delta(B, B'; \Omega)$ and $\beta \in \Delta(B', \partial\Omega; \Omega)$.

Now let $\rho \in \mathrm{adm}(A, B; \Omega)$, and for $\lambda > 0$ consider the set

$$A_\lambda = \{\zeta \in A : \text{there is } r = r_\zeta,\ 0 < r \leq 4ad, \text{ with } \int_{\Omega\cap B(\zeta;r)} \rho^Q\,d\mu \geq \lambda(r/2ad)^p\}.$$

We claim that there exists a $\lambda_0 > 0$, depending only on the data, such that $A = A_{\lambda_0}$. Given this claim, we finish the proof as follows. Appealing to a basic covering theorem (cf. [Hei01, Theorem 1.2]) we find points $\zeta_i \in A$ with associated radii $r_i = r_{\zeta_i}$ such that the balls $B_i = B(\zeta_i; r_i)$ are pairwise disjoint and $A \subset \cup_i 5B_i$. Then we deduce that

$$\int_\Omega \rho^Q\,d\mu \geq \sum_i \int_{\Omega\cap B_i} \rho^Q\,d\mu \geq \frac{\lambda_0}{(2ad)^p} \sum_i r_i^p \geq \frac{\lambda_0}{(20ad)^p}\mathcal{H}^p(A) \geq \frac{c\,\lambda_0}{(20a)^p};$$

here we have used $\mathrm{diam}(5B_i) \leq 10r_i$ along with the hypothesis that $\mathcal{H}^p(A) \geq c\,d^p$. The above gives $\mathrm{mod}_Q(B, A; \Omega) \geq m = c\,\lambda_0/(10a)^p > 0$ and m depends only on the data associated with Ω.

Thus it suffices to establish the above claim. Fix $\lambda > 0$. Suppose $A \neq A_\lambda$ and let $\zeta \in A \setminus A_\lambda$. Thus

$$\int_{\Omega\cap B(\zeta;r)} \rho^Q\,d\mu < \lambda\left(\frac{r}{2ad}\right)^p \qquad \text{for all } 0 < r \leq 4ad.$$

We construct a locally rectifiable curve $\sigma \in \Delta(B, A; \Omega)$ with $\int_\sigma \rho\,ds \leq C\,\lambda^{1/Q}$, which gives $\lambda \geq \lambda_0 = 1/C^Q$ as desired.

First we note that for all points $z \in [x, \zeta)$,

$$d(z) \leq |z - \zeta| \leq \ell[x, \zeta) \leq a|x - \zeta| \leq 2ad,$$

and if also $z \notin B$, then in addition we have

$$|z - \zeta| \leq 5a\,d(z).$$

The first string of inequalities is clear, as is the second inequality when z is 'closer' to ζ; if z is 'closer' to x and not in B, then $d/2 \leq |x - z| \leq \ell[x, z] \leq a\,d(z)$ which yields

$$|z - \zeta| \leq |z - x| + |x - \zeta| \leq a\,d(z) + 2d \leq 5a\,d(z)$$

as desired.

Appealing to Fact 3.1 we obtain successive points z_i along $[x,\zeta)$ satisfying

$$k(z_i,z_{i+1}) = \varepsilon_0/12 \quad \text{(for all } i\text{) and} \quad \sum_i d_i^{p/Q} \le C_1 L^{p/Q},$$

where $d_i = d(z_i)$, $L = \ell[x,\zeta)$, and $C_1 = C_1(p,Q,a,\varepsilon_0)$. To simplify our notation, we assume that z_0 is the last of the z_i's in B; thus $z_0 \in B$ and $z_i \notin B$ for all $i \ge 1$. As in [BHK01, (6.24),(6.25),(6.26),p.53], we find that

$$\frac{\varepsilon_0 d_i}{96a^2} \le |z_i - z_j| \le \frac{\varepsilon_0 d_i}{3} \quad (i \ge 0, j = i+1 \text{ or } i \ge 1, j = i-1);$$
$$\frac{d_i}{2} \le d_{i-1} \le 2d_i \quad (i \ge 1);$$
$$\frac{\varepsilon_0 d_i}{1000a^2} \le \operatorname{diam}(\gamma_i) \le \frac{\varepsilon_0 d_i}{500a^2} \quad (i \ge 1).$$

Here γ_i are subarcs of $[x,\zeta)$ which join z_i to $S(z_i;\varepsilon_0 d_i/1000a^2)$. Put $B_i = B(z_i;\varepsilon_0 d_i)$ and notice that $\gamma_{i+1} \subset B_i$ for all $i \ge 0$.

Now $d/2 < |x - z_1| \le |x - z_0| + \varepsilon_0 d_0/3 \le |x - z_0| + \varepsilon_0 d/2$, so $|x - z_0| \ge d_0/6$ and we can select a subarc $\beta_0 \subset [x,z_0]$ with $z_0 \in \beta_0 \subset B_0$ and

$$\operatorname{diam}(\beta_0) \ge \varepsilon_0 d_0 \ge \frac{\varepsilon_0}{4a} d(z) \quad \text{for any } z \in [x,\zeta).$$

Since $\operatorname{dist}(\beta_0,\gamma_1) \le |z_0 - z_1| \le \varepsilon_0 d_0/3$, we see that

$$\Sigma(\beta_0,\gamma_1) \le 1000a^2 = C_2, \quad \text{which implies that} \quad \operatorname{mod}_Q(\beta_0,\gamma_1;\kappa B_0) \ge m = \psi(C_2) > 0.$$

This lower bound on the modulus means that there must exist a locally rectifiable arc $\alpha_0 \in \Delta(\beta_0,\gamma_1;\kappa B_0)$ with

$$\int_{\alpha_0} \rho\, ds < \left(\frac{\lambda}{m}\right)^{1/Q} \left(\frac{5d_0}{d}\right)^{p/Q}.$$

Indeed, if not, then ρ divided by right-hand-side of the above inequality would belong to $\operatorname{adm}(\beta_0,\gamma_1;\kappa B_0)$, which would yield the contradiction

$$m \le \frac{m}{\lambda}\left(\frac{d}{5d_0}\right)^p \int_{\Omega \cap B(\zeta;2|\zeta - z_0|)} \rho^Q\, d\mu < m;$$

here we have used the fact that $\kappa B_0 \subset B(z_0;d_0) \subset \Omega \cap B(\zeta;2|\zeta - z_0|)$—which holds because $d_0 \le |z_0 - \zeta| \le 2ad$—along with the information that $\zeta \notin A_\lambda$, which means that

$$\int_{\Omega \cap B(\zeta;2|\zeta - z_0|)} \rho^Q\, d\mu < \lambda \left(\frac{|\zeta - z_0|}{ad}\right)^p,$$

and finally that $|z_0 - \zeta| \le 5a\, d_0$.

Let x_1 be the endpoint of α_0 on γ_1. Choose a subcontinuum $\beta_1 \subset \alpha_0 \cup \beta_0$ with $x_1 \in \beta_1 \subset B_1$ and

$$\frac{\varepsilon_0}{1000a^2} d_1 \le \operatorname{diam}(\beta_1) \le \frac{\varepsilon_0}{500a^2} d_1.$$

(Since $\operatorname{diam}(\beta_0) \ge d_2/4a$, such a β_1 clearly exists when $\alpha_0 \cup \beta_0 \subset B_1$; otherwise, when $(\alpha_0 \cup \beta_0) \setminus B_1 \ne \emptyset$, we use the fact that the x_1-component of $(\alpha_0 \cup \beta_0) \cap \bar{B}_1$ must meet ∂B_1; cf. [Kur68, Theorem 1,p.172] or [HY88, Theorem 2-16,p.47]). Then $\beta_1, \gamma_2 \subset B_1$ with

$\Sigma(\beta_1,\gamma_2) \le C_2$, so $\operatorname{mod}_Q(\beta_1,\gamma_2;\kappa B_1) \ge m$. As above we obtain a locally rectifiable arc $\alpha_1 \in \Delta(\beta_1,\gamma_2;\kappa B_1)$ with

$$\int_{\alpha_1} \rho\, ds < \left(\frac{\lambda}{m}\right)^{1/Q} \left(\frac{5d_1}{d}\right)^{p/Q}.$$

Let us check that we can continue this construction. To this end, suppose that we have selected locally rectifiable arcs $\alpha_0,\dots,\alpha_n$ and continua $\beta_0,\dots,\beta_n$ with $\alpha_i \in \Delta(\beta_i,\gamma_{i+1};\kappa B_i)$ and $x_i \in \beta_i \subset B_i \cap (\beta_0 \cup \alpha_0 \cup \dots \cup \alpha_{i-1})$, where x_i is the endpoint of α_{i-1} on γ_i. We claim that there is a continuum β_{n+1} satisfying $x_{n+1} \in \beta_{n+1} \subset B_{n+1} \cap (\beta_0 \cup \alpha_0 \cup \dots \cup \alpha_n)$ and

$$\frac{\varepsilon_0}{1000a^2} d_{n+1} \le \operatorname{diam}(\beta_{n+1}) \le \frac{\varepsilon_0}{500a^2} d_{n+1}.$$

The crucial point here is that $\beta_0 \cup \alpha_0 \cup \dots \cup \alpha_n$ is a continuum. Then either it lies entirely inside the ball B_{n+1}, in which case we can find such a β_{n+1} because β_0 is large, or it leaves this ball in which case the x_{n+1}-component of $(\beta_0 \cup \alpha_0 \cup \dots \cup \alpha_n) \cap \bar{B}_{n+1}$ must meet ∂B_{n+1} (by [Kur68, Theorem 1,p.172] or [HY88, Theorem 2-16,p.47]).

Then $\beta_{n+1},\gamma_{n+2} \subset B_{n+1}$ with $\Sigma(\beta_{n+1},\gamma_{n+2}) \le C_2$, so $\operatorname{mod}_Q(\beta_{n+1},\gamma_{n+2};\kappa B_1) \ge m$. Thus as above we obtain a locally rectifiable arc $\alpha_{n+1} \in \Delta(\beta_{n+1},\gamma_{n+2};\kappa B_{n+1})$ with

$$\int_{\alpha_{n+1}} \rho\, ds < \left(\frac{\lambda}{m}\right)^{1/Q} \left(\frac{5d_{n+1}}{d}\right)^{p/Q}.$$

It is easy to see that $\cup_i \alpha_i$ contains a locally rectifiable curve $\sigma \in \Delta(B,A;\Omega)$, and so

$$\begin{aligned} 1 \le \int_\sigma \rho\, ds \le \sum_i \int_{\alpha_i} \rho\, ds &\le \left(\frac{\lambda}{m}\right)^{1/Q} \left(\frac{5}{d}\right)^{p/Q} \sum_i d_i^{p/Q} \le \\ &\le C_1 \left(\frac{\lambda}{m}\right)^{1/Q} \left(\frac{5}{d}\right)^{p/Q} L^{p/Q} \le C_1 (10a)^{p/Q} \left(\frac{\lambda}{m}\right)^{1/Q}, \end{aligned}$$

where we have used the fact that $L = \ell[x,\zeta) \le a|x-\zeta| \le 2ad$. We now see that

$$\lambda \ge \lambda_0 = m/[(10a)^p C_1^Q],\ m = \psi(1000a^2),\ C_1 = C_1(p,Q,a,\varepsilon_0).$$

◇

3.D. **Remark.** In Theorem F, it is necessary to assume that Ω and $\partial\Omega$ are simultaneously bounded or unbounded. For example, the domain $\Omega = \mathbf{R}^n \setminus \bar{\mathbf{B}}^n$ has $\partial\Omega = \mathbf{S}^{n-1}$ certainly has positive $(n-1)$-density, but Ω fails to possess WBMP.

3.E. **Proof of Theorem G.** Let $A \subset \bar{\Omega}$ be compact, (Ω,d,μ) an a-uniform locally Q-Loewner metric measure space, and suppose $\mathcal{H}^p(A) > 0$ for some $p > 0$. We verify that A does not have zero Q-modulus. (Recall the definition given at the end of subsection 2.F.) We may assume that $A \subset B(P;1)$ for some point $P \in \Omega$. Choose $\zeta \in \partial\Omega$ so that $d(P) = |P-\zeta|$. We consider two cases: $d(P) \ge 2$, $d(P) < 2$.

Suppose that $R = d(P)/2 \ge 1$. Fix a point $x \in [P,\zeta)$ with $|x-P| = 3R/2$ and put $B = \bar{B}(x;d(x)/10)$. Then

$$R/2 \le d(x) \le |x-\zeta| \le 4R$$

and so $A \cap B = \emptyset$. We show that $\operatorname{mod}_Q(B,A;\Omega) > 0$.

First, let $v \in \bar{B}(P;R)$ (and note that $B(P;R) \supset B(P;1) \supset A$). Then

$$R/2 \le |x-v| \le 3R;$$

thus for any $z \in [x, v]$ we find that

$$|z - v| \le 3aR,$$

and if in addition $z \notin B$, then also

$$|z - v| \le 61a\, d(z).$$

The first of the above two inequalities is clear, as is the second when z is 'closer' to v; if z is 'closer' to x and not in B, then $d(x)/10 \le |x - z| \le \ell[x, z] \le a\, d(z)$ which yields

$$|z - v| \le |z - x| + |x - v| \le a\, d(z) + 3R \le a(z) + 6d(x) \le 61a\, d(z)$$

as desired. In particular we see that

$$B(z; d(z)) \subset B(v; |z - v| + d(z)) \subset B(v; 62a\, d(z)).$$

Now let $\rho \in \operatorname{adm}(A, B; \Omega)$, and for $\lambda > 0$ consider the set

$$A_\lambda = \{v \in A : \text{there is } r = r_v,\ 0 < r \le 9aR, \text{ with } \int_{\Omega\cap B(v;r)} \rho^Q\, d\mu \ge \lambda r^p\}.$$

We claim that there exists a $\lambda_0 > 0$, depending only on the data, such that $A = A_{\lambda_0}$. Given this claim, we finish the proof as follows. Appealing to a basic covering theorem (cf. [Hei01, Theorem 1.2]) we find points $v_i \in A$ with associated radii $r_i = r_{v_i}$ such that the balls $B_i = B(v_i; r_i)$ are pairwise disjoint and $A \subset \cup_i 5B_i$. Then we deduce that

$$\int_\Omega \rho^Q\, d\mu \ge \sum_i \int_{\Omega\cap B_i} \rho^Q\, d\mu \ge \lambda_0 \sum_i r_i^p \ge \frac{\lambda_0}{10^p}\mathcal{H}^p(A) > 0;$$

here we have used $\operatorname{diam}(5B_i) \le 10r_i$ along with the hypothesis that $\mathcal{H}^p(A) > 0$. Since the above lower bound does not depend on ρ, we deduce that $\operatorname{mod}_Q(B, A; \Omega) > 0$, which means that A does not have zero Q-modulus.

Thus it suffices to establish the above claim. Fix $\lambda > 0$. Suppose $A \ne A_\lambda$ and let $v \in A \setminus A_\lambda$. Thus

$$\int_{\Omega\cap B(v;r)} \rho^Q\, d\mu < \lambda r^p \qquad \text{for all } 0 < r \le 9aR.$$

We construct a locally rectifiable curve $\sigma \in \Delta(B, A; \Omega)$ with $\int_\sigma \rho\, ds \le C\, \lambda^{1/Q}$, which gives $\lambda \ge \lambda_0 = 1/C^Q$ as desired.

Employing (2.1) we obtain

$$k(x, v) \ge \log(1 + |x - v|/d(x)) \ge \log(9/8);$$

since $\log(9/8) < 1/8$, we can appeal to Fact 3.1, with $\varepsilon_1 = \min\{\varepsilon_0/12, \log(9/8)\}$, to obtain successive points z_i along $[x, v]$. As in the above proof (and in the proof of [BHK01, Theorem 6.4]) we use these points to select subarcs γ_i of $[x, z]$ and then find continua β_i with

$$\operatorname{mod}_Q(\beta_i, \gamma_{i+1}; \kappa B_i) \ge m = \psi(C(a)).$$

This modulus inequality guarantees the existence of locally rectifiable arcs $\alpha_i \in \Delta(\beta_i, \gamma_{i+1}; \kappa B_i)$ with

$$\int_{\alpha_i} \rho\, ds < \left(\frac{\lambda}{m}\right)^{1/Q} (62a\, d_i)^{p/Q}.$$

Here $B_i = B(z_i; \varepsilon_0 d_i)$, $d_i = d(z_i)$, and the existence of α_i is a consequence of the inequality

$$\int_{\kappa B_i} \rho^Q\, d\mu \le \int_{\Omega\cap B(v;|z_i - v| + d_i)} \rho^Q\, d\mu < \lambda(62ad_i)^p;$$

this holds because $v \notin A_\lambda$, $\kappa B_i \subset B(z_i; d_i) \subset \Omega \cap B(v; |z_i - v| + d_i)$ and $|z_i - v| \leq 61a\, d_i$.

Now $\cup_i \alpha_i$ contains a locally rectifiable curve $\sigma \in \Delta(B, A; \Omega)$, and so

$$1 \leq \int_\sigma \rho\, ds \leq \sum_i \int_{\alpha_i} \rho\, ds \leq \left(\frac{\lambda}{m}\right)^{1/Q} (62a)^{p/Q} \sum_i d_i^{p/Q} \leq$$
$$\leq C_1 \left(\frac{\lambda}{m}\right)^{1/Q} (62a)^{p/Q}\, L^{p/Q} \leq C_1 (186aR)^{p/Q} \left(\frac{\lambda}{m}\right)^{1/Q},$$

where we have used the fact that $L = \ell[x, v] \leq 3R$. We now see that

$$\lambda \geq \lambda_0 = m/[(186aR)^p C_1^Q],\ m = \psi(C(a)),\ C_1 = C_1(p, Q, a, \varepsilon_0).$$

Now suppose that $d(P) = |P - \zeta| < 2$. Let $\Omega' = \Omega \cap B(P; 2)$, choose a point $w \in \Omega'$ with

$$d(w) \geq \frac{1}{2} \sup_{x \in \Omega'} d(x),$$

and put $R = d(w)/9a$. Note that $d(w) \leq 4$, so $aR \leq 4/9$ and $d(x) \leq 8$ for all $x \in \Omega'$. Since A is compact with $\mathcal{H}^p(A) > 0$, we can further assume that $A \subset B(y; R)$ for some point $y \in \Omega \cap B(P; 1)$.

Next we select a point $u \in \Omega$ with $|u - y| \geq 4aR$. (If $|w - y| \geq 4aR$ we can take $u = w$; otherwise choose a point $u \in \Omega \cap B(\xi; aR)$ where $\xi \in \partial\Omega$ satisfies $d(w) = |w - \xi|$.) Then $\ell[u, y] \geq 4aR$, so we can pick $x \in [u, y]$ satisfying $\ell[x, y] = 2aR$. It follows that

$$2R \leq |x - y| \leq 2aR \quad \text{and} \quad 2R \leq d(x) \leq 18aR;$$

the very last inequality holds because $x \in \Omega'$ (since $|x - P| \leq |x - y| + |y - P| < 2aR + 1 < 2$) and so $d(x) \leq 2d(w)$. Put $B = \bar{B}(x; d(x)/20a)$ and note that $A \cap B = \emptyset$. We show that $\operatorname{mod}_Q(B, A; \Omega) > 0$.

First, let $v \in \bar{B}(y; R)$ (and note that $B(y; R) \supset A$). Then

$$|x - v| \leq 3aR;$$

thus for any $z \in [x, v]$ we find that

$$|z - v| \leq 3a^2 R,$$

and if in addition $z \notin B$, then also

$$|z - v| \leq 31a^3 d(z).$$

The first of the above two inequalities is clear, as is the second when z is 'closer' to v; if z is 'closer' to x and not in B, then $d(x)/20a \leq |x - z| \leq \ell[x, z] \leq a\, d(z)$ which yields

$$|z - v| \leq |z - x| + |x - v| \leq a\, d(z) + 3aR \leq a(z) + (3a/2)d(x) \leq 31a^3 d(z)$$

as desired. In particular, $B(z; d(z)) \subset B(v; |z - v| + d(z)) \subset B(v; 32a^3\, d(z))$.

Now let $\rho \in \operatorname{adm}(A, B; \Omega)$, and for $\lambda > 0$ consider the set

$$A_\lambda = \{v \in A : \text{there is } r = r_v,\ 0 < r \leq 7a, \text{ with } \int_{\Omega \cap B(v; r)} \rho^Q\, d\mu \geq \lambda r^p\}.$$

As above, it suffices to produce a $\lambda_0 > 0$, depending only on the data, such that $A = A_{\lambda_0}$; this ensures that A does not have zero Q-modulus.

Fix $\lambda > 0$, suppose $A \neq A_\lambda$ and let $v \in A \setminus A_\lambda$. Thus

$$\int_{\Omega\cap B(v;r)} \rho^Q \, d\mu < \lambda r^p \qquad \text{for all } 0 < r \leq 7a.$$

We construct a locally rectifiable curve $\sigma \in \Delta(B, A; \Omega)$ with $\int_\sigma \rho \, ds \leq C\,\lambda^{1/Q}$, which gives $\lambda \geq \lambda_0 = 1/C^Q$ as desired.

Employing (2.1) we obtain

$$k(x, v) \geq \log(1 + |x - v|/d(x)) \geq \log(1 + 1/(18a));$$

since $\log(1+1/(18a)) \leq \log(19/18) < 1/8$, we can appeal to Fact 3.1, with $\varepsilon_2 = \min\{\varepsilon_0/12, \log(1+1/(18a))\}$, to obtain successive points z_i along $[x, v]$. As above, we use these points to select subarcs γ_i of $[x, z]$ and then find continua β_i with

$$\operatorname{mod}_Q(\beta_i, \gamma_{i+1}; \kappa B_i) \geq m = \psi(C(a)).$$

This modulus inequality guarantees the existence of locally rectifiable arcs $\alpha_i \in \Delta(\beta_i, \gamma_{i+1}; \kappa B_i)$ with

$$\int_{\alpha_1} \rho \, ds < \left(\frac{\lambda}{m}\right)^{1/Q} \left(36a^3\, d_i\right)^{p/Q}.$$

Here $B_i = B(z_i; \varepsilon_0 d_i)$, $d_i = d(z_i)$, and the existence of α_i is a consequence of the inequality

$$\int_{\kappa B_i} \rho^Q \, d\mu \leq \int_{\Omega\cap B(v;|z_i - v| + d_i)} \rho^Q \, d\mu < \lambda(32a^3 d_i)^p;$$

this holds because $v \notin A_\lambda$, $\kappa B_i \subset B(z_i; d_i) \subset \Omega \cap B(v; |z_i - v| + d_i)$ and $|z_i - v| \leq 31a^3\, d_i$.

Now $\cup_i \alpha_i$ contains a locally rectifiable curve $\sigma \in \Delta(B, A; \Omega)$, and so

$$1 \leq \int_\sigma \rho \, ds \leq \sum_i \int_{\alpha_i} \rho \, ds \leq \left(\frac{\lambda}{m}\right)^{1/Q} \left(32a^3\right)^{p/Q} \sum_i d_i^{p/Q} \leq$$

$$\leq C_1 \left(\frac{\lambda}{m}\right)^{1/Q} \left(32a^3\right)^{p/Q} L^{p/Q} \leq C_1 (64a^4)^{p/Q} \left(\frac{\lambda}{m}\right)^{1/Q},$$

where we have used the fact that $L = \ell[x, v] \leq a|x - v| \leq 3a^2 R = a\, d(w)/3 \leq 2a$. We now see that

$$\lambda \geq \lambda_0 = m/[(64a^4)^p C_1^Q], \ m = \psi(C(a)), \ C_1 = C_1(p, Q, a, \varepsilon_0).$$

$\diamond$

4. The Gehring Hayman Inequality

In this section we prove Theorem A.

4.A. **Preliminaries.** The following two results provide the essential estimates needed in the main argument; these key ideas appear imbedded in the proofs of [BKR98, Theorem 3.1] and [HR93, Theorem 1.1]

4.1. Lemma. *Let (Ω, d) be an a-uniform metric space. Assume that ρ is a continuous density on Ω satisfying* (HI). *Suppose that γ is any arc which meets some quasihyperbolic geodesic $[x, y]$ and has length $|x - y| \leq b\,\ell(\gamma)$. Then*

$$\ell_\rho[x, y] \leq C\,\ell_\rho(\gamma),$$

where $C = C(b, a, H, K)$, $H = H(\rho)$, $K = k(x, y)$.

Proof. Fix a point $z \in \gamma \cap [x,y]$. Since one of the subarcs of $\gamma \setminus \{z\}$ has length at least $|x-y|/2b$, we may as well assume that z is an endpoint of γ. Appealing to Fact 2.3 we obtain

$$\ell_\rho[x,y] \le H_1 \rho(z) \ell[x,y] \le a H_1 \rho(z) |x-y|,$$

where $H_1 = H^{1+3K}$.

Suppose that $k(z,w) \le K$ for all $w \in \gamma$. Utilizing Fact 2.3 again we find that

$$\ell_\rho(\gamma) \ge \frac{\rho(z)}{H_1} \ell(\gamma) \ge \frac{\rho(z)|x-y|}{bH_1},$$

and thus

$$\ell_\rho[x,y] \le C\, \ell_\rho(\gamma) \qquad \text{where } C = abH_1^2.$$

Assume that there is some point of γ with quasihyperbolic distance to z greater than K, and let w be the first point along γ with $k(z,w) = K$. We show that $|z-w| \ge c\,|x-y|$, which then allows us to apply the above reasoning to the arc $\gamma(z,w)$. First note that

$$K = k(x,y) \ge j(x,y) \implies |x-y| \le c_1 \min\{d(x), d(y)\},\ c_1 = e^K - 1.$$

Similar inequalities hold for $|x-z|$, $|y-z|$, $|w-z|$. Next,

$$K = k(z,w) \le a\, j(z,w) \implies |z-w| \ge c_2 \min\{d(z), d(w)\},\ c_2 = e^{K/a} - 1.$$

Now

$$\min\{d(z), d(w)\} \ge \frac{d(z)}{1+c_1} \ge \frac{\max\{|x-z|, |y-z|\}}{c_1(1+c_1)} \ge \frac{|x-y|}{2c_1(1+c_1)},$$

so $|z-w| \ge |x-y|/b_1$ and $\ell(\gamma(z,w)) \ge |x-y|/b_1$ where $b_1 = 2c_1(1+c_1)/c_2$. Applying the first part of the proof we conclude that

$$\ell_\rho[x,y] \le C_1\, \ell_\rho(\gamma) \qquad \text{where } C_1 = ab_1 H_1^2.$$

$\diamond$

4.2. Lemma. *Let (Ω, d, μ) be an a-uniform Q-Loewner metric measure space. Assume that ρ is a conformal density on Ω satisfying both* (HI), (VG). *Suppose that γ is any arc with*

$$b \operatorname{diam}(\gamma) \ge \operatorname{diam}[x,y] \ge \frac{1}{b} \operatorname{dist}([x,y], \gamma).$$

Then

$$\ell_\rho[x,y] \le C\, \ell_\rho(\gamma),$$

where $C = C(b, a, H, K, M, Q, \varphi)$, $H = H(\rho)$, $M = M(\rho)$, $K = k(x,y)$, and φ is the Loewner control function.

Proof. If γ meets $[x,y]$, we can appeal to Lemma 4.1. Assume $\gamma \cap [x,y] = \emptyset$. Then we have a pair of non-degenerate disjoint continua with

$$\Sigma([x,y], \gamma) = \frac{\operatorname{dist}([x,y], \gamma)}{\min\{\operatorname{diam}[x,y], \operatorname{diam}(\gamma)\}} \le b^2,$$

so we know that $\operatorname{mod}_Q([x,y], \gamma) \ge \varphi(b^2) = c_1$.

Let $c > 1$ and suppose that $\ell_\rho(\sigma) \ge c\, \ell_\rho(\gamma)$ for all locally rectifiable arcs $\sigma \in \Delta([x,y], \gamma)$. The Basic Modulus Estimate (Fact 2.8) then yields

$$c_1 \le \operatorname{mod}_Q([x,y], \gamma) \le c_2 [\log(1+c)]^{1-Q} \quad (\text{where } c_2 = 2^{Q+1} M / \log 2).$$

Thus any such c must satisfy $c \le \exp[(c_1/c_2)^{1/(Q-1)}] - 1$, so letting $c_3 = c_3(b, M, Q, \varphi) = \exp[(c_1/c_2)^{1/(Q-1)}]$ we deduce the existence of some arc $\sigma \in \Delta([x,y],\gamma)$ with $\ell_\rho(\sigma) \le c_3\ell_\rho(\gamma)$.

Next, according to Lemma 4.1, there is a constant $c_4 = c_4(b, a, H, K)$ with the property that $\ell_\rho[x,y] \le c_4\ell_\rho(\alpha)$ for any arc α which meets $[x,y]$ and has $\ell(\alpha) \ge |x-y|/2b$. Now if $\ell_\rho[x,y] \le 2c_4\ell_\rho(\sigma)$, then we find that $\ell_\rho[x,y] \le 2c_3c_4\ell_\rho(\gamma)$ and we are done. Thus we may assume that $\ell_\rho[x,y] \ge 2c_4\ell_\rho(\sigma)$.

Finally, let z, w be the endpoints of σ with $z \in [x,y]$, $w \in \gamma$. Choose $v \in \gamma$ so that $|w-v| \ge \operatorname{diam}(\gamma)/2 \ge |x-y|/2b$. Consider $\alpha = \sigma \cup \tau$ where $\tau = \gamma(w,v)$: this is an arc which meets $[x,y]$ with $\ell(\alpha) \ge \ell(\tau) \ge |x-y|/2b$. Thus

$$\ell_\rho[x,y] \le c_4\ell_\rho(\alpha) \le \frac{1}{2}\ell_\rho[x,y] + c_4\ell_\rho(\tau)$$

and so we obtain $\ell_\rho[x,y] \le 2c_4\ell_\rho(\gamma)$. ◇

Having disposed with the above preliminaries, we now proceed to the main event.

4.B. **Proof of Theorem A.** Assume that (Ω, d, μ) is a Q-Loewner a-uniform metric measure space and that ρ is a conformal density on Ω with Harnack constant $H = H(\rho)$ and volume growth constant $M = M(\rho)$. Let $[x,y]$ be a quasihyperbolic geodesic, and let γ be any other arc with the same endpoints x, y in $\bar\Omega$. We verify that $\ell_\rho[x,y] \le C\ell_\rho(\gamma)$ which establishes the theorem.

Order both arcs 'from x to y'. We consider separately the cases where both endpoints belong to Ω, both belong to $\partial\Omega$, or one endpoint is in Ω and one on $\partial\Omega$. In the first case our argument follows the lines of [HR93, Theorem 1.1] with minor modifications, in the second we follow [BKR98, Theorem 3.1], and a combination of the two works for the final case.

The case $x, y \in \Omega$. Assume that $d(x) \le d(y)$. We consider three subcases depending on whether $|x-y| \le 4d(x)$ or $4d(x) < |x-y| \le 4d(y)$ or $4d(y) < |x-y|$.

Suppose $|x-y| \le 4d(x)$. Thanks to Fact 2.2 we easily find that $k(x,y) \le K(a) = 4a^2\log 5$; an appeal to Lemma 4.1 now yields $\ell_\rho[x,y] \le C\ell_\rho(\gamma)$ with $C = C(a,H)$.

Suppose that $4d(x) < |x-y| \le 4d(y)$. Let $m \ge 0$ be the integer satisfying

$$2^m d(x) < \frac{|x-y|}{4} \le 2^{m+1}d(x).$$

Note that both arcs $[x,y]$, γ leave the ball $B(x; 2^{m+2}d(x))$. We consider an expanding sequence of concentric spherical rings centered at x and associated subarcs of $[x,y]$, γ.

Let $x_{-1} = x$, $x_{m+1} = y$ and for $0 \le i \le m$ let x_i be the first point along $[x,y]$ with $|x_i - x| = 2^i d(x)$. Thus $[x,y] = \cup[x_{i-1}, x_i]$. Next let $\alpha_0 = \gamma(x, u_0)$, $\alpha_{m+1} = \gamma(u_m, y)$ where u_0, u_m are the first, last points along γ with $|u_0 - x| = d(x)$, $|u_m - x| = 2^m d(x)$. Then for $1 \le i \le m$ let α_i be a subarc of γ which joins the spheres $S(x; 2^i d(x))$, $S(x; 2^{i-1}d(x))$ inside the ring $B(x; 2^i d(x)) \setminus \bar B(x; 2^{i-1}d(x))$. We show that for each $0 \le i \le m+1$,

$$\ell_\rho[x_{i-1}, x_i] \le C\ell_\rho(\alpha_i);$$

since the x_i are successive points along $[x,y]$ and the α_i are non-overlapping subarcs of γ, the desired inequality follows by summing the above inequalities.

We begin by pointing out that

$$d(z) \ge 2^{i-1}d(x)/a \qquad \text{for all } z \in [x_{i-1}, x_i] \text{ and all } 1 \le i \le m.$$

Indeed, for any such point z we have

$$\ell[x,z] \geq |x-x_{i-1}| = 2^{i-1}d(x) \text{ and } \ell[y,z] \geq |y-x_m| \geq 3\cdot 2^m d(x),$$

so the above follows because $[x,y]$ is an a-uniform arc. Next we use Fact 2.2 to deduce that

$$k(x_{i-1},x_i) \leq K = K(a) = 4a^2\log(1+8a^2) \qquad \text{for all } 0 \leq i \leq m+1.$$

Now $[x,x_0]$, α_0 both have endpoint x, and $\ell(\alpha_0) \geq d(x) = |x-x_0|$, so Lemma 4.1 provides the estimate $\ell_\rho[x,x_0] \leq C\ell_\rho(\alpha_0)$ with $C = C(a,H)$.

We turn our attention to the arcs $[x_{i-1},x_i]$, α_i for $1 \leq i \leq m$. Since these both have an endpoint on the sphere $S(x;2^{i-1}d(x))$, we find that

$$\operatorname{dist}([x_{i-1},x_i],\alpha_i) \leq 2^i d(x) \leq 2\operatorname{diam}[x_{i-1},x_i] \leq 2^{i+2}d(x) \leq 8\operatorname{diam}(\alpha_i).$$

Lemma 4.2 permits us to assert that $\ell_\rho[x_{i-1},x_i] \leq C\ell_\rho(\alpha_i)$ with $C = C(a,H,M,Q,\varphi)$.

Next, both subarcs $[x_m,y]$, α_{m+1} have endpoint y, and $\ell(\alpha_{m+1}) \geq 3\cdot 2^m d(x) \geq (3/8)|x-y| \geq (3/8a)|x_m-y|$, so Lemma 4.1 gives $\ell_\rho[x_m,y] \leq C\ell_\rho(\alpha_{m+1})$ with $C = C(a,H)$.

We examine the remaining subcase where $4d(y) < |x-y|$. As above we get integers $m \geq 0$, $n \geq 0$ satisfying

$$2^m d(x) < \frac{|x-y|}{4} \leq 2^{m+1}d(x), \qquad 2^n d(y) < \frac{|x-y|}{4} \leq 2^{n+1}d(y).$$

Also as above we select successive points $x_{-1} = x, x_0, \dots, x_m$ and $y_{-1} = y, y_0, \dots, y_n$ along $[x,y]$ and non-overlapping subarcs α_i, β_j of γ, where $0 \leq i \leq m$, $0 \leq j \leq n$. Arguing as above we obtain $\ell_\rho[x_{i-1},x_i] \leq C\ell_\rho(\alpha_i)$ and $\ell_\rho[y_{j-1},y_j] \leq C\ell_\rho(\beta_j)$. Thus it remains to demonstrate that $\ell_\rho[x_m,x_n] \leq C\ell_\rho(\gamma_0)$ where $\gamma_0 = \gamma(u_m,v_n)$ and u_m (resp., v_n) is the first (last) point along γ with $|u_m - x| = 2^m d(x)$ (resp., $|v_n - y| = 2^n d(y)$).

Our choices of m,n ensure that $|x-y| > 2^{m+1}d(x) + 2^{n+1}d(y)$, so $B(x;2^{m+1}d(x)) \cap B(y;2^{n+1}d(y)) = \emptyset$. In particular then,

$$\min\{\operatorname{diam}[x_m,y_n],\operatorname{diam}(\gamma_0)\} \geq 2^m d(x) + 2^n d(y).$$

Also,

$$\begin{aligned}\operatorname{dist}([x_m,y_n],\gamma_0) \leq \min\{|x_m-u_m|,|y_n-v_n|\} &\leq \min\{2^{m+1}d(x),2^{n+1}d(y)\} \leq \\ &\leq 2\operatorname{diam}[x_m,y_n] \leq 2a|x-y| \leq 8a\operatorname{diam}(\gamma_0).\end{aligned}$$

Next, $d(x_m) \geq (1/a)\min\{|x-x_m|,|y-x_m|\} \geq (2^m/a)d(x) \geq |x-y|/8a$ and similarly $d(y) \geq |x-y|/8a$; hence $k(x_m,y_n) \leq 4a^2 j(x_m,y_n) \leq 4a^2\log(1+8a^2)$. An appeal to Lemma 4.2 produces the necessary inequality $\ell_\rho[x_m,x_n] \leq C\ell_\rho(\gamma_0)$ with $C = C(a,H,M,Q,\varphi)$.

The case $x,y \in \partial\Omega$. Put $d = |x-y|/4$ and let x_i (resp., y_j) be the first (last) point along $[x,y]$ on the sphere $S(x;d/2^i)$ (resp., $S(y;d/2^j)$). Then $[x,x_i) \subset B(x;d/2^i)$, $[y,y_j) \subset B(y;d/2^j)$, and

$$(x,y) = \bigcup_{i=0}^{\infty}[x_i,x_{i+1}] \cup [x_0,y_0] \cup \bigcup_{j=0}^{\infty}[y_j,y_{j+1}]$$

with the above an essentially disjoint (i.e., non-overlapping) union.

Set $\gamma_0 = \gamma(u_0,v_0)$ where u_0, v_0 are the last, first points along γ on the spheres $S(x;d)$, $S(y;d)$ respectively. Next, choose subarcs α_i, β_j of γ which join the spheres $S(x;d/2^i)$ to

$S(x;d/2^{i+1})$, $S(y;d/2^j)$ to $S(y;d/2^{j+1})$ inside the rings $B(x;d/2^i)\setminus\bar{B}(x;d/2^{i+1})$, $B(y;d/2^j)\setminus\bar{B}(y;2^{j+1})$ respectively. Then

$$\bigcup_{i=0}^{\infty}\alpha_i\cup\gamma_0\cup\bigcup_{j=0}^{\infty}\beta_j\subset\gamma$$

and this is a non-overlapping union.

Arguing as before, we can employ Lemmas 4.1 and 4.2 to verify that

$$\ell_\rho[x_0,y_0]\le C\ell_\rho(\gamma_0),\quad \ell_\rho[x_i,x_{i+1}]\le C\ell_\rho(\alpha_i),\quad \ell_\rho[y_j,y_{j+1}]\le C\ell_\rho(\beta_j).$$

Summing these inequalities gives the desired result.

The case $x\in\Omega$, $y\in\partial\Omega$. Again, let $d=|x-y|/4$ and choose points y_j and subarcs $\beta_j\subset\gamma$ exactly as above. From above, $\ell_\rho[y_j,y_{j+1}]\le C\ell_\rho(\beta_j)$. In the subcase $d(x)\ge d$, it is straightforward to use Lemma 4.1 to see that $\ell_\rho[x,y_0]\le C\ell_\rho(\gamma(x,v_0)$ (where v_0 is as above), and so the desired inequality follows. Let us examine the final subcase, $d(x)<d$.

As at the very beginning of this proof, we let $m\ge 0$ be the integer satisfying

$$2^m d(x)<d\le 2^{m+1}d(x).$$

Set $x_{-1}=x$ and for $0\le i\le m$ let x_i be the first point along $[x,y]$ with $|x_i-x|=2^i d(x)$. Thus

$$[x,x_i)\subset B(x;2^i d(x))\quad\text{and}\quad [x,y]=\bigcup_{i=0}^{m}[x_{i-1},x_i]\cup[x_m,y_0]\cup\bigcup_{j=0}^{\infty}[y_j,y_{j+1}].$$

Next let $\alpha_0=\gamma(x,u_0)$, $\gamma_0=\gamma(u_m,v_0)$ where u_0, u_m, v_0 are the first, last, last points along γ with $|u_0-x|=d(x)$, $|u_m-x|=2^m d(x)$, $|v_0-y|=d$ respectively. Then for $1\le i\le m$ let α_i be a subarc of γ which joins the spheres $S(x;2^i d(x))$, $S(x;2^{i-1}d(x))$ inside the ring $B(x;2^i d(x))\setminus\bar{B}(x;2^{i-1}d(x))$. As in the first part of this proof, for each $0\le i\le m$,

$$\ell_\rho[x_{i-1},x_i]\le C\ell_\rho(\alpha_i)\quad\text{and}\quad \ell_\rho[x_m,y_0]\le C\ell_\rho(\gamma_0).$$

Summing these inequalities along with the corresponding ones, $\ell_\rho[y_j,y_{j+1}]\le C\ell_\rho(\beta_j)$, we obtain the desired inequality $\ell_\rho[x,y]\le C\ell_\rho(\gamma)$.

$\diamond$

4.C. **Application.** Recall that we let Λ designate the Gehring-Hayman Inequality constant. Combining the Gehring-Hayman Inequality with Lemma 2.4 we arrive at the following complement to Corollary 2.5.

4.3. Corollary. *Let ρ be a conformal density on an a-uniform Loewner metric measure space (Ω,d,μ). Then for all $z\in\Omega$,*

$$d_\rho(x,y)\ge\frac{1}{\Lambda H}\rho(z)|x-y|\quad\textit{for all } x,y\in B(z;d(z)/4a),$$

where $H=H(\rho)$. In particular, $\operatorname{diam}_\rho B(z;d(z)/4a)\ge 4a\Lambda H\rho(z)d(z)$.

5. Boundary Behavior

Here we analyze the behavior of a conformal density near the space's boundary; of course the Harnack Inequality asserts that any conformal density is essentially constant on compact subspaces. In particular, we establish Theorems B, C and D.

5.A. **Proof of Theorem B.** Let ρ be a conformal density on a uniform Q-Loewner metric measure space (Ω, d, μ). Fix a point $w \in \Omega$. We begin by showing that the set A, of points $\zeta \in \partial\Omega$ with $\ell_\rho(\gamma) \leq L$ for some geodesic ray γ from w to ζ, is closed in $\bar{\Omega}$. (Note that the metric completion of a locally compact metric space need not be locally compact, for example see [BH99, (4),p.34], so some care is required here.) Let $\{\zeta_i\}$ be a sequence of points in A which converges to ζ in $\bar{\Omega}$. Since $\partial\Omega$ is closed in $\bar{\Omega}$, $\zeta \in \partial\Omega$.

Let γ_i be geodesic rays from w to ζ_i with $\ell_\rho(\gamma_i) \leq L$ for all i. Let $x_i(t)$ be quasihyperbolic arclength parameterizations for γ_i. Then $\{x_i\}$ forms an equicontinuous family of maps from $[0, \infty)$ into (Ω, k) which is pointwise bounded (e.g. since $k(w, x_i(t)) = t$). Since closed bounded subsets of (Ω, k) are compact [BHK01, Proposition 2.8], a general theorem of Ascoli's [Mun00, 47.1, p.290 & 46.2, p.283] asserts the existence of a subsequence $\{y_j\}$ of $\{x_i\}$ which converges uniformly on compact subsets to some function $y : [0, \infty) \to (\Omega, k)$. From the basic inequality (2.1) we easily deduce that we have uniform convergence (on compacta) for (Ω, d) as well.

We claim that $\gamma = y([0, \infty))$ is a quasihyperbolic geodesic from w to ζ with $\ell_\rho(\gamma) \leq L$, so $\zeta \in A$. Consider any compact subarc $\alpha = y([0, \tau])$ of γ. Since y_j converges uniformly to y on $[0, \tau]$ and

$$\ell(\gamma_j) \leq a|w - \zeta_j| \leq 2a|w - \zeta| \qquad \text{for all (large) } j,$$

we see that α is rectifiable and $\ell(\alpha) \leq a|w - \zeta|$ (cf. [BH99, 1.23, p.14 & (7), p.13]. Thus γ is rectifiable with $\ell(\gamma) \leq a|w - \zeta|$. Clearly γ has endpoints w and ζ.

Again by uniform convergence we find that

$$k(y(s), y(t)) = \lim_{j\to\infty} k(y_j(s), y_j(t)) = |s - t|.$$

In other words, $y : [0, \infty) \to (\Omega, k)$ is an isometric embedding; that is, γ is a geodesic ray. Finally, writing ds (resp., ds_k) for the d (resp., k) element of arclength, and noticing that $ds_k(z) = ds(z)/d(z)$, we find that

$$\ell_\rho(\alpha) = \int_0^\tau \rho(y(t))\, ds(y(t)) = \int_0^\tau \rho(y(t))d(y(t))\, ds_k(y(t)) =$$
$$= \int_0^\tau \rho(y(t))d(y(t))\, dt = \lim_{j\to\infty} \int_0^\tau \rho(y_j(t))d(y_j(t))\, dt \leq \limsup_{j\to\infty} \ell_\rho(\gamma_j) \leq L$$

as desired.

The proof now proceeds as in [BKR98, Theorem 4.4]. For each positive integer m we let A_m be the set of points $\zeta \in \partial\Omega$ with $\ell_\rho(\gamma) \leq m$ for some geodesic ray γ from w to ζ. (Note that, according to the Gehring-Hayman Inequality, for a fixed point $\zeta \in \partial\Omega$, $\ell_\rho(\gamma)$ is either finite or infinite for all geodesic rays going to ζ.) Since A_m is closed,

$$A = \{\zeta \in \partial\Omega : \ell_\rho[w, \zeta) = \infty\} = \partial\Omega \setminus \cup_1^\infty A_m$$

is a Borel set. It remains to explain why $\operatorname{mod}_Q A = 0$.

Put $B = \bar{B}(w; d(w)/2)$, $\lambda = \operatorname{diam}_\rho B$ and let $\Gamma = \Delta(B, A; \Omega)$. Then $\ell_\rho(\gamma) = \infty$ for all $\gamma \in \Gamma$. According to the Basic Modulus Estimate (Fact 2.8), for all $L > \lambda$ we have

$$\operatorname{mod}_Q \Gamma \leq c/\log(1 + L/\lambda)^{Q-1};$$

letting L tend to ∞ produces the asserted conclusion. $\diamond$

5.B. **Applications.** Theorem B proved above asserts that for conformal deformations of a uniform Loewner space, 'radial limits' exist 'mod_Q-a.e.' Consequently, so do 'angular limits', in the sense described below.

Fix $w \in \Omega$, $\zeta \in \partial\Omega$ and let γ be any curve in Ω joining w to ζ. For $0 < \lambda < 1$ and $0 < \delta \le |w-\zeta|$ we define the twisted Stolz cone at ζ (with core γ) by

$$\mathrm{cone}_w(\gamma;\lambda,\delta) = \bigcup_{x\in\gamma(z,\zeta)} B(x;\lambda\, d(x)),$$

where z is the first point along γ on the sphere $S(\zeta;\delta)$. Next, writing $\Gamma_w(\zeta)$ for the family of quasihyperbolic geodesic rays from w to ζ, we define a 'non-tangential approach region' to ζ via

$$\mathrm{Cone}_w(\zeta;\lambda,\delta) = \bigcup_{\gamma\in\Gamma_w(\zeta)} \mathrm{cone}_w(\gamma;\lambda,\delta).$$

Also, let $G_w(\zeta) = \{x \in \Omega : x \in [w,\zeta) \quad \text{for some quasihyperbolic geodesic ray}\}$ (i.e., the union of all the rays in $\Gamma_w(\zeta)$) and finally,

$$G_w(\zeta;\delta) = \bigcup_{\gamma\in\Gamma_w(\zeta)} \gamma(z,\zeta) = \{x\in\Omega : x\in\gamma(z,\zeta) \quad \text{for some } \gamma\in\Gamma_w(\zeta)\},$$

where (again) z is the first point along γ on the sphere $S(\zeta;\delta)$. (Notice that for such a point z,

$$d(z) \le |z-\zeta| = \delta \le a\, d(z)$$

with the last inequality holding for sufficiently small δ when Ω is a-uniform and γ a quasihyperbolic geodesic ray going to ζ.)

5.1. Lemma. *If ρ is a continuous density on an a-uniform space Ω and ρ satisfies* (HI)*, then for each $0<\lambda<1$ there is a constant $C = C(\lambda,a,H)$ such that*

$$\mathrm{diam}_\rho \mathrm{Cone}_w(\zeta,\lambda,\delta) \le C\, \mathrm{diam}_\rho G_w(\zeta;\delta)$$

for all $\zeta\in\partial\Omega$ and $0<\delta\le|w-\zeta|$.

Proof. Fix $\zeta \in \partial\Omega$, $0<\delta\le|w-\zeta|$, and let $\gamma\in\Gamma_w(\zeta)$. We begin by showing that $d_\rho(x,y) \le C\,\mathrm{diam}_\rho\gamma(z,\zeta)$ whenever $x\in\gamma(z,\zeta)$ and $y\in\bar B(x;\lambda\, d(x))$, where z is the first point along γ on the sphere $S(\zeta;\delta)$. Since $d(y)\ge(1-\lambda)d(x)$, Fact 2.2 gives us

$$k(x,y) \le 4a^2 j(x,y) \le 4a^2\log(1/(1-\lambda)) = K(\lambda,a) = K.$$

Thus by Fact 2.3 and Lemma 4.1 we obtain

$$\begin{aligned} d_\rho(x,y) &\le \ell_\rho[x,y] \le H_1\rho(x)\ell[x,y] \le aH_1\rho(x)|x-y| \le aH_1\rho(x)|x-u| \\ &\le aH_1^2\ell_\rho[x,u] \le aH_1^2C_1 d_\rho(x,u) \le C\,\mathrm{diam}_\rho\gamma(z,\zeta) \end{aligned}$$

where $H_1 = H^{3K+1}$, $C_1 = C_1(\lambda,a,H)$ and u is any point of $\gamma(z,\zeta)\cap S(x;\lambda\, d(x))$.

Now given $y_1,y_2\in\mathrm{Cone}_w(\zeta,\lambda,\delta)$, choose $\gamma_1,\gamma_2\in\Gamma_w(\zeta)$ and $x_i\in\gamma_i\cap S(\zeta;\delta)$ so that $y_i\in\bar B(x_i;\lambda\, d(x_i))$. Then

$$d_\rho(y_1,y_2) \le d_\rho(x_1,y_1) + d_\rho(x_1,x_2) + d_\rho(x_2,y_2) \le C\,\mathrm{diam}_\rho G_w(\zeta;\delta)$$

as desired. $\diamond$

An upshot of the following, in conjunction with Theorem B, is that conformal deformations of uniform Loewner spaces possess 'angular limits' 'mod_Q-a.e.'.

5.2. Corollary. *Let ρ be a conformal density on a uniform Loewner space Ω. Fix $w \in \Omega$ and suppose $\zeta \in \partial\Omega$ is d_ρ-rectifiably accessible. Then for each $0 < \lambda < 1$,*

$$\lim_{\delta \to 0} \operatorname{diam}_\rho \operatorname{Cone}_w(\zeta; \lambda, \delta) = 0.$$

Proof. According to Lemma 5.1, it suffices to establish that $\lim_{\delta\to 0} \operatorname{diam}_\rho G_w(\zeta;\delta) = 0$ for each d_ρ-rectifiably accessible boundary point ζ. This in turn is a consequence of the fact that the 'tails' of any two quasihyperbolic geodesic rays in $\Gamma_w(\zeta)$ will have comparable ρ-length. We explain the details.

Let $\alpha, \beta \in \Gamma_w(\zeta)$ and choose $x \in \alpha$, $y \in \beta$ with $|x - \zeta| = \delta = |y - \zeta|$. Then, provided δ is small enough, $\min\{d(x), d(y)\} \geq \delta/a$, so $k(x, y) \leq K(a)$ (here Ω is a-uniform). Thus Lemma 4.1 gives us $\ell_\rho[x, y] \leq C\ell_\rho(\beta)$ with $C = C(a, H)$. Next, the Gehring-Hayman Inequality applies to produce

$$\ell_\rho(\alpha) \leq \Lambda\, \ell_\rho([x, y] \cup \beta) \leq (C + 1)\Lambda\, \ell_\rho(\beta).$$

$\diamond$

The following growth estimate for a conformal density, valid for most geodesic rays, is another consequence of the above results. Note that we also have pointwise estimates when our space is regular; see Theorem 7.2.

5.3. Theorem. *Let ρ be a conformal density on a uniform locally Q-Loewner metric measure space (Ω, d, μ). For each $0 < p \leq \dim_{\mathcal{H}}\partial\Omega$, there exists a set $E \subset \partial\Omega$ with $\mathcal{H}^p(E) = 0$ and such that for all $\zeta \in \partial\Omega \setminus E$,*

$$\rho(z) = o(\delta^{(p/Q)-1}) \qquad \text{as } \delta \to 0,$$

uniformly for $z \in S(\zeta;\delta) \cap G_w(\zeta)$ where w is any fixed point of Ω.

Proof. As in the proof of Theorem B, we let A_j be the set of points $\zeta \in \partial\Omega$ with $\ell_\rho(\gamma) \leq j$ for some geodesic ray γ from w to ζ. Recall that $A = \partial\Omega \setminus \cup A_j$ has Q-modulus zero, hence $\dim_{\mathcal{H}}(A) = 0$ by Theorem G. (This is the only place we require the local Loewner condition.) Our desired set E will consist of A plus a countable union of small sets.

Put

$$B_j = \bigcup_{\zeta \in A_j} \operatorname{Cone}_w(\zeta; 1/2, |w - \zeta|).$$

Using Lemma 5.1 and the proof of Corollary 5.2 we see that $R = \operatorname{diam}_\rho B_j \leq C\, j$; thus $B_j \subset B_\rho(w; R)$ and therefore $\mu_\rho(B_j) < \infty$. Hence $f_j = \rho^Q \chi_{B_j}$ is μ-integrable over Ω. (Here χ_S denotes the characteristic function of a set S.) Lemma 5.4 below now furnishes a set $E_j \subset \partial\Omega$ with $\mathcal{H}^p(E_j) = 0$ and such that for all $\zeta \in \partial\Omega \setminus E_j$,

$$\int_{\Omega \cap B(\zeta;r)} f_j \, d\mu = o(r^p) \qquad \text{as } r \to 0.$$

Let $\zeta \in A_j \setminus E_j$ and $z \in S(\zeta;\delta) \cap G_w(\zeta)$. Then $B(z; d(z)/2) \subset B(\zeta; (3/2)\delta)$, so

$$\int_{B(z;d(z)/2)} \rho^Q \, d\mu \leq \int_{\Omega \cap B(\zeta;(3/2)\delta)} f_j \, d\mu = o(\delta^p) \qquad \text{as } \delta \to 0.$$

Using (HI) and the fact that μ must satisfy a lower mass condition [HK98, (3.7), p.15], say $\mu B(x; r) \geq c r^Q$ for all $x \in \Omega$ and all $0 < r < d(x)$ (with $c = c(Q, \varphi)$), we see that

$$H^Q \int_{B(z;d(z)/2)} \rho^Q \, d\mu \geq c\, \rho(z)^Q\, (d(z)/2)^Q$$

and thus

$$\rho(z)^Q \leq C\delta^{-Q} o(\delta^p) \qquad \text{as } \delta \to 0,$$

where $C = C(a, H, Q, \varphi)$.

In particular, we infer that

$$\rho(z) = o(\delta^{(p/Q)-1}) \qquad \text{as } \delta \to 0,$$

and this holds for all points $z \in S(\zeta;\delta) \cap G_w(\zeta)$ when $\zeta \in A_j \setminus E_j$. Letting $E = A \cup \bigcup E_j$, we see that $\mathcal{H}^p(E) = 0$ and the desired estimate holds for all points $\zeta \in \partial\Omega \setminus E$. $\diamond$

For the reader's convenience, we include a proof of the following.

5.4. Lemma. *Let f be a non-negative integrable function on a non-complete locally compact metric measure space (Ω, d, μ). Then for all $p > 0$, there exists a set $E \subset \partial\Omega$ with $\mathcal{H}^p(E) = 0$ and such that for all points $\zeta \in \partial\Omega \setminus E$,*

$$\int_{\Omega \cap B(\zeta;r)} f\, d\mu = o(r^p) \qquad \text{as } r \to 0.$$

Proof. For each positive integer m we let

$$A_m = \{\xi \in \partial\Omega : \limsup_{r\to 0} r^{-p} \int_{\Omega\cap B(\xi;r)} f\, d\mu > 1/m\}.$$

Then $E = \cup A_m$ is our desired set, so it suffices to show that each set $A = A_m$ has zero p-content. For $i = 1, 2, \dots$ we consider the family $\mathcal{F}_i$ of all balls $B(\xi;r)$ with $\xi \in A$, $0 < r < 1/i$, and $\int_{\Omega\cap B(\xi;r)} f\, d\mu \geq (1/m)r^p$. Each $\mathcal{F}_i$ is a cover of A, so according to a basic covering theorem (cf. [Hei01, Theorem 1.2]) there are disjoint balls $B_j = B(\xi_j; r_j)$ in $\mathcal{F}_i$ such that $A \subset \bigcup_j 5\, B_j$. Thus

$$\mathcal{H}^p(A) \leq 10^p \sum_j r_j^p \leq m\, 10^p \sum_j \int_{\Omega\cap B_j} f\, d\mu \leq m\, 10^p \int_{\{x\in\Omega: d(x)<1/i\}} f\, d\mu.$$

Since f is μ-integrable, $\int_{\{x\in\Omega:d(x)<1/i\}} f\, d\mu \to 0$ as $i \to \infty$. Therefore $\mathcal{H}^p(A) = 0$. $\diamond$

5.C. **Proof of Theorem C.** Let (Ω, d, μ) be a bounded a-uniform Q-Loewner metric measure space with Loewner control function φ. Assume that ρ is a conformal density on Ω satisfying both (HI), (VG) and let E be any continuum in Ω. Choose $w \in \Omega$ with $d(w) = \sup\{d(x) : x \in \Omega\}$ and put $B = \bar{B}(w; d(w)/20a)$. Note that

$$1 \leq \operatorname{diam}(\Omega)/d(w) \leq 2a.$$

As in [BKR98, Lemma 4.1], we proceed to estimate $\operatorname{diam}(E)$ in terms of $\operatorname{diam}_\rho(E)$.

If $\operatorname{diam}_\rho(E) \geq b\, \rho(w)d(w)$ for some constant b, then

$$\frac{\operatorname{diam}(E)}{\operatorname{diam}(\Omega)} \leq \frac{1}{b}\frac{\operatorname{diam}_\rho(E)}{\rho(w)d(w)}.$$

Assume that $\operatorname{diam}_\rho(E) \leq \rho(w)d(w)/6aH$. First, suppose $E \cap B \neq \emptyset$. In this case, $E \subset B(w; d(w)/4a)$—because any curve joining B to points outside this latter ball has a subarc α with

$$\ell_\rho(\alpha) \geq \frac{\rho(w)}{H}\ell(\alpha) \geq \frac{\rho(w)d(w)}{5aH}$$

which cannot be a lower bound for $\operatorname{diam}_\rho(E)$—and thus by Theorem A and Lemma 2.4,

$$\rho(w)|x - y| \leq \Lambda H d_\rho(x, y) \leq \Lambda H \operatorname{diam}_\rho(E)$$

for all $x, y \in E$. It now follows that

$$\frac{\operatorname{diam}(E)}{\operatorname{diam}(\Omega)} \leq \frac{\Lambda H}{2} \frac{\operatorname{diam}_\rho(E)}{\rho(w)d(w)}.$$

The above reasoning continues to hold when $\operatorname{diam}_\rho(E)$ is even smaller, so we are left with the (main) case when

$$\operatorname{diam}_\rho(E) < \frac{\rho(w)d(w)}{40aH}, \quad E \cap B = \emptyset, \quad B = \bar{B}(w; d(w)/20a).$$

Let $F = (1/2)B = \bar{B}(w; d(w)/40a)$. Then every (locally rectifiable) curve γ joining E, F has a subarc joining F to ∂B and thus satisfies

$$\ell_\rho(\gamma) \geq \frac{\rho(w)d(w)}{40aH} = L > \lambda = \operatorname{diam}_\rho(E).$$

An appeal to the Basic Modulus Estimate (Fact 2.8) gives us

$$\operatorname{mod}_Q(E, F; \Omega) \leq c \left(\log \left(1 + \frac{\rho(w)d(w)}{40aH \operatorname{diam}_\rho(E)} \right) \right)^{1-Q}$$

where $c = 2^{Q+1}M/\log 2$, $M = M(\rho)$.

On the other hand, $\operatorname{diam}(F) \geq d(w)/40a$, so

$$\operatorname{diam}(E) \leq \operatorname{diam}(\Omega) \leq 2a\, d(w) \leq 80a^2 \operatorname{diam}(F)$$

which gives $\Sigma(E, F) \leq 80a^2 \operatorname{diam}(\Omega)/\operatorname{diam}(E)$. Therefore by the Loewner property we deduce that

$$\varphi\left(80a^2 \frac{\operatorname{diam}\Omega}{\operatorname{diam}E}\right) \leq \operatorname{mod}_Q(E, F; \Omega) \leq c \left(\log \left(1 + \frac{\rho(w)d(w)}{40a)H \operatorname{diam}_\rho(E)} \right) \right)^{1-Q}.$$

In this setting we must have $\lim_{t\to\infty} \varphi(t) = 0$ (see Lemma 2.10), so we can define $\varphi^{-1}(s) = \inf \varphi^{-1}(0, s]$ and $\lim_{s\to 0} \varphi^{-1}(s) = \infty$. Using φ^{-1} and the above inequalities we now see how to construct a function ψ with the asserted properties. When $\varphi(t) \geq b(\log t)^{1-Q}$ for large t and some constant b (which is true if Ω is Q-regular by [HK98, Theorem 3.6, (3.11)]) we easily find that $\psi(s) = Cs^\alpha$ with $\alpha = (b/c)^{1/(Q-1)}$ and $C = C(b, a, H, M, Q, \varphi)$.

To see that $\alpha \leq 1$: Select continua E, F with $\operatorname{diam}(F) \geq \operatorname{diam}(E) \geq \varepsilon\, r$, $E \subset \bar{B}(z; \varepsilon\, r)$ and $F \cap B(z; r) = \emptyset$ where $r = d(z)/4a$ and $\varepsilon = 1/(1+t)$ with t large. Then $\Sigma(E, F) \leq t$, and each $\gamma \in \Delta(E, F; \Omega)$ has $\ell_\rho(\gamma) \geq (1-\varepsilon)r\rho(z)/H = L > \lambda = 2\varepsilon\, raH\rho(z) \geq \operatorname{diam}_\rho(E)$ (see Lemma 2.4). Thus the Loewner condition in conjunction with the Basic Modulus Estimate yields

$$b(\log t)^{1-Q} \leq \varphi(t) \leq \operatorname{mod}_Q(E, F; \Omega) \leq c \left(\log \left(1 + \frac{\rho(z)(1-\varepsilon)r}{2aH^2\varepsilon r} \right) \right)^{1-Q}$$

which can only hold for large $t = (1-\varepsilon)/\varepsilon$ if $b \leq c$, so $\alpha = (b/c)^{1/(Q-1)} \leq 1$. $\diamond$

5.D. **Remarks.** Below we provide a simple example to illustrate the need for assuming Ω bounded in Theorem C. In subsection 6.A we employ Theorem C to realize $\partial_\rho\Omega$ as a subspace of $\partial\Omega$; this fails to hold in the example below.

5.5. Example. *The identity map $id : \Omega_\rho \to \Omega$ may not be continuous.*

Proof. Take Ω to be the upper-half-plane in $\mathbf{R}^2$, which we identify with the complex plane $\mathbf{C}$, and let

$$\rho(z) = \left|\frac{dw}{dz}\right| = \frac{2}{|z+i|^2} \qquad \text{where} \qquad w = \frac{z-i}{z+i}.$$

It is easy to check that as $0 < x \to \infty$,

$$|(x+i) - (2x+i)| \to \infty \qquad \text{whereas} \qquad d_\rho(x+i, 2x+i) \to 0.$$

Moreover, if we let E_x denote the horizontal Euclidean segment $\{t+i : x \le t \le 2x\}$, then as $x \to \infty$, $\operatorname{diam}(E_x) \to \infty$ but $\operatorname{diam}_\rho(E_x) \to 0$. $\diamond$

However, when Ω is bounded, we have the following upshot of Theorem C.

5.6. Corollary. *For a conformal density ρ on a bounded uniform Loewner metric measure space Ω,*

$$|x-y| \le \omega(d_\rho(x,y)) \qquad \textit{for all } x,y \in \Omega,$$

where the modulus of continuity function is $\omega(t) = \operatorname{diam}(\Omega)\psi(\Lambda\, t)$.

Also, in the special case when Ω is Q-regular, we even have Hölder continuity of $id : \Omega_\rho \to \Omega$ with exponent $\alpha = (b/c)^{1/(Q-1)}$; see the end of the proof of Theorem C and also Section 7.

5.E. **Proof of Theorem D.** Let (Ω, d, μ) be a bounded a-uniform locally Q-Loewner metric measure space with Loewner control function φ. Fix a point $w \in \Omega$. Assume that ρ is a conformal density on Ω satisfying both (HI), (VG). Suppose that $\{x_k\}$ is a Cauchy sequence in Ω_ρ which converges in $\bar{\Omega}$ to some point $\xi \in \partial\Omega$. We construct a d_ρ-rectifiable curve in Ω joining w to ξ. Then the Gehring-Hayman Inequality asserts that $\ell_\rho[w,\xi] \le \Lambda d_\rho(x,\xi) < \infty$. We proceed as in [BKR98, Theorem 4.2].

Since $\{x_k\}$ is d_ρ-Cauchy, we can extract a subsequence $\{z_j\}$ with $d_\rho(z_j, z_{j+1}) \le 2^{-j}$ for all j. Select arcs β_j joining z_j, z_{j+1} with

$$\ell_\rho(\beta_j) \le 2 d_\rho(z_j, z_{j+1}) \le 2^{1-j},$$

and let β_0 be any arc joining w to z_1 with $\ell_\rho(\beta_0) < \infty$. Clearly then $\beta = \cup \beta_j$ is a d_ρ-rectifiable curve joining w to ξ, but before we can utilize the Gehring-Hayman Inequality we must know that β joins these points in Ω; i.e., that β has endpoint ξ as a curve in Ω.

Let $z = z(t)$, $0 \le t < 1$ be a parametrization for β with $z(0) = w$. Choose $0 < t_1 < \cdots < t_j \to 1$ with $z(t_j) = z_j$. For $0 \le t < 1$ consider the curve $F_t = \{z(s) : t \le s < 1\}$. Since $t_j \to 1$ and $\sum_j^\infty \ell_\rho(\beta_i) \le 2^{-j}$, we see that $\operatorname{diam}_\rho(F_t) \to 0$ as $t \to 1$. Appealing to Theorem C (applied to compact subcurves of F_t) we deduce that $\operatorname{diam}(F_t) \to 0$ as $t \to 1$ as well. This ensures that β has an endpoint on $\partial\Omega$, and since $z_j \to \xi$, we must have $\xi = \lim_{t\to 1} z(t)$. Now the Gehring-Hayman Inequality gives us

$$\ell_\rho[w,\xi] \le \Lambda d_\rho(w,\xi) \le \Lambda \ell_\rho(\beta) < \infty$$

as desired.

Next let $\{y_k\}$ be any sequence along $[w,\xi)$ which converges to ξ. It remains to verify that $d_\rho(x_k, y_k) \to 0$. We continue with the notation introduced above. Let $E_t = [w(t), \xi)$ where $w(t)$, $0 \le t < 1$ is an injective parametrization for $[w,\xi)$. Then $\operatorname{diam}_\rho(E_t) \le \ell_\rho[w(t),\xi) \to 0$ as $t \to 1$.

We have two non-degenerate, closed, connected sets $E_t, F_t \subset \Omega$ with $\bar{E}_t \cap \bar{F}_t \ne \emptyset$. According to Fact 2.12, $\operatorname{mod}_Q(E_t, F_t; \Omega) = \infty$. (We point out that this is the only place where we use the 'extra' hypothesis that Ω be locally Loewner.) This means that there are arcs $\alpha_t \in \Delta(E_t, F_t; \Omega)$ with $\ell_\rho(\alpha_t) < 2\,\mathrm{diam}_\rho(E_t)$ (for otherwise we could use the Basic Modulus Estimate to obtain a finite upper bound for $\mathrm{mod}_Q(E_t, F_t; \Omega)$). Thus we see that

$$\mathrm{diam}_\rho(E_t \cup F_t) \le 3\,\mathrm{diam}_\rho(E_t) + \mathrm{diam}_\rho(F_t) \to 0 \quad \text{as } t \to 1.$$

Since E_t, F_t contain the 'tails' of the sequences $\{y_j\}$, $\{z_j\}$ respectively, we see that $d_\rho(y_j, z_j) \to 0$, and since $\{z_j\}$ is a subsequence of the d_ρ-Cauchy sequence $\{x_k\}$, $d_\rho(y_k, x_k) \to 0$ also. $\diamond$

6. Geometry of Ω_ρ

Here we study the deformed space $\Omega_\rho = (\Omega, d_\rho, \mu_\rho)$. We begin by describing its boundary $\partial_\rho\Omega = \bar{\Omega}_\rho \setminus \Omega$. Then we estimate $d_\rho(x) = \mathrm{dist}_\rho(x, \partial\Omega)$ and explain how this shows that (Ω_ρ, k_ρ) is bilipschitz equivalent to (Ω, k). We also establish a separation property of Ω_ρ.

6.A. **Remarks about $\partial_\rho\Omega$.** Simple examples reveal that it is possible to have $\partial_\rho\Omega = \emptyset \neq \partial\Omega$; e.g. for $\Omega = \mathbf{R}^2 \setminus \{0\}$ and $\rho(x) = 1/|x|$ we see that Ω_ρ is the cylinder $\mathbf{S}^1 \times \mathbf{R}$.

On general principles we know that $\partial_\rho\Omega$ contains the set F_ρ of points in $\partial\Omega$ which are d_ρ-rectifiably accessible, i.e.,

$$\partial_\rho\Omega \supset F_\rho = \{\zeta \in \partial\Omega : \ell_\rho[w, \zeta) < \infty \quad \text{for some } w \in \Omega\}.$$

(Given a point $\zeta \in F_\rho$ one easily constructs a d_ρ-Cauchy sequence which converges to ζ in $\bar{\Omega}_\rho$.) Thus according to Theorems B and G $\dim_{\mathcal{H}}(\partial_\rho\Omega) \ge \dim_{\mathcal{H}}(\partial\Omega)$ (here in both cases $\dim_{\mathcal{H}}$ means that the Hausdorff measures are defined via the original distance on Ω). In particular, $\partial_\rho\Omega \neq \emptyset$ if $\dim_{\mathcal{H}}(\Omega) > 0$ (and Ω is uniform and locally Loewner).

Of course $\partial_\rho\Omega$ need not lie inside $\partial\Omega$; cf. Example 5.5. However, when Ω is bounded we have Theorem C which ensures that the identity map $id : \Omega_\rho \to \Omega$ is uniformly continuous. In this situation, every Cauchy sequence in Ω_ρ will be Cauchy in Ω too, so we have a natural inclusion $\bar{\Omega}_\rho \subset \bar{\Omega}$ and especially $\partial_\rho\Omega \subset \partial\Omega$. Also, $\dim_{\mathcal{H}}(\partial_\rho\Omega) = \dim_{\mathcal{H}}(\partial\Omega)$ and even $\dim_{\mathcal{H}}(\partial\Omega \setminus \partial_\rho\Omega) = 0$ thanks to Theorems B and G (again, if Ω is uniform and locally Loewner).

In fact, it is straightforward, using Theorem D, to verify the following.

6.1. Corollary. *Let ρ be a conformal density on a bounded uniform locally Loewner metric measure space (Ω, d, μ). Then*

$$\partial_\rho\Omega = F_\rho = \{\zeta \in \partial\Omega : \ell_\rho[w, \zeta) < \infty \quad \textit{for some } w \in \Omega\}.$$

Our primary interest here is in obtaining estimates for $d_\rho(x) = \mathrm{dist}_\rho(x, \partial_\rho\Omega)$ in terms of $\rho(x)d(x)$. The Harnack inequality supplies an immediate lower bound. To obtain an upper bound, we at least need $\partial_\rho\Omega \neq \emptyset$. As we just saw, this will be the case if $\dim_{\mathcal{H}}(\partial\Omega) > 0$. However, even then we may find that $d_\rho(x) \to \infty$ while $\rho(x)d(x)$ stays bounded; e.g., taking $\Omega = \mathbf{B}^2 \setminus \{0\}$ and $\rho(x) = 1/|x|$ we see that Ω_ρ is the semi-infinite cylinder $\mathbf{S}^1 \times [0, \infty)$ and $\rho(x)d(x) = 1$ as $x \to 0$ while $d_\rho(x) = \log(1/|x|) \to \infty$.

The example above illustrates the need for a condition which will ensure that Ω has a boundary which is 'uniformly thick at all scales'. Thus we are naturally led to a condition like the Whitney ball modulus property as given in the Introduction and analyzed in subsection 3.B.

6.B. **Proof of Theorem E.** Let (Ω, d, μ) be an a-uniform Q-Loewner metric measure space with Loewner control function φ. Suppose that Ω satisfies the Whitney ball modulus property. Assume that ρ is a conformal density on Ω satisfying both (HI), (VG).

Fix a point $z \in \Omega$ and put $B = \bar{B}(z; d(z)/4a)$. The Comparison Principle (Fact 2.6) can be used to see that

$$\operatorname{mod}_Q(B, \partial\Omega; \Omega) \geq q = q(m, a, Q, \varphi) > 0.$$

Indeed,

$$\operatorname{mod}_Q(B, \partial\Omega; \Omega) \geq 3^{-Q} \min\{\operatorname{mod}(B, S; \Omega), \operatorname{mod}_Q(S, \partial\Omega; \Omega), I\},$$

where

$$I = \inf\{\operatorname{mod}_Q(\alpha, \beta; \Omega) : \alpha \in \Delta(B, S; \Omega), \beta \in \Delta(S, \partial\Omega; \Omega)\}$$

and $S = \bar{B}(z; d(z)/2) \backslash B(z; d(z)/3)$. Now $\operatorname{mod}_Q(S, \partial\Omega; \Omega) = \operatorname{mod}_Q(\bar{B}(z; d(z)/2), \partial\Omega; \Omega) \geq m$ by hypothesis. Next, by examining continua in B, S we find that $\operatorname{mod}_Q(B, S; \Omega) \geq \varphi(4a)$. Similarly, $I \geq \varphi(6)$.

According to Corollary 2.5 we have

$$\operatorname{diam}_\rho(B) \leq H\rho(z)d(z)/2.$$

Next, if $\gamma \in \Delta(B, \partial\Omega; \Omega)$ joins a point $x \in B$ to $\partial\Omega$, then $d_\rho(z) \leq d_\rho(z, x) + d_\rho(x) \leq \operatorname{diam}_\rho(B) + \ell_\rho(\gamma)$. We conclude that

$$\ell_\rho(\gamma) \geq L = d_\rho(z) - \operatorname{diam}_\rho(B) \qquad \text{for all } \gamma \in \Delta(B, \partial\Omega; \Omega).$$

Thus if $L \leq \operatorname{diam}_\rho(B)$, then $d_\rho(z) \leq 2\operatorname{diam}_\rho(B) \leq H\rho(z)d(z)$. On the other hand, if $L > \operatorname{diam}_\rho(B)$, then the Basic Modulus Estimate (Fact 2.8) gives us

$$q \leq \operatorname{mod}_Q(B, \partial\Omega; \Omega) \leq c\left(\log(L/\operatorname{diam}_\rho(B)\right)^{1-Q}$$

so $L \leq \exp\left((c/q)^{1/(Q-1)}\right)\operatorname{diam}_\rho(B)$ and we conclude that

$$d_\rho(z) \leq L + \operatorname{diam}_\rho(B) \leq bH\rho(z)d(z)$$

where $b = b(m, a, M, Q, \varphi) = 2[1 + \exp\left((c/q)^{1/(Q-1)}\right)]$. $\diamond$

6.C. **Quasihyperbolization of Ω_ρ.** We consider the quasihyperbolic distance k_ρ in Ω_ρ obtained from d_ρ. Of course,

$$k_\rho(x, y) = \inf \int_\gamma \frac{ds_\rho}{d_\rho(z)}$$

where the infimum is taken over all d_ρ-rectifiable curves γ which join x, y. Since $ds_\rho = \rho(z)ds$, Theorem E easily gives us

$$\frac{1}{bH} \leq \frac{k_\rho(x, y)}{k(x, y)} \leq 2H \qquad \text{for all } x, y \in \Omega.$$

We conclude that (Ω, k_ρ) and (Ω, k) are bilipschitz equivalent whenever ρ is a conformal density on a uniform Loewner space Ω which satisfies a Whitney ball modulus property. In particular, since (Ω, k) is Gromov hyperbolic [BHK01, Theorem 3.6], so is (Ω, k_ρ) (see [BBI01, Theorem 8.4.16]).

While it is true that many properties of Ω_ρ can now be deduced from this fact that (Ω, k_ρ) is Gromov hyperbolic, one should remember that this is a consequence of Theorem E which in turn relies heavily on the Gehring-Hayman Inequality (and of course the Whitney ball modulus property). In particular, it is known that if Ω is a domain in $\mathbf{R}^n$ and (Ω, k) is Gromov hyperbolic, then both the Gehring-Hayman Inequality and a certain Separation

Property hold [BHK01, p.75]. In fact, recently Balogh and Buckley [BBpp] have corroborated that these conditions characterize Gromov hyperbolicity for a general class of metric spaces.

6.D. **Proof of Theorem H.** Here we demonstrate that the above mentioned Separation Property holds for Ω_ρ when ρ is a conformal density (satisfying both (HI), (VG)) on an a-uniform Q-Loewner metric measure space (Ω, d, μ) (with Loewner control function φ) which satisfies the Whitney ball modulus property.

Fix points $w \in \Omega$, $\zeta \in \partial\Omega$ and $x \in [w, \zeta)$. Put $B = \bar{B}_\rho(x; d_\rho(x)/2)$. Let A be any arc joining $[w, x]$ to $[x, \zeta)$. We confirm that $d_\rho(x, A) \leq C d_\rho(x)$. Assume that $\mathrm{dist}_\rho(x, A) \geq 3d_\rho(x)/2$. Then every curve $\gamma \in \Delta(A, B; \Omega)$ has

$$\ell_\rho(\gamma) \geq L = \mathrm{dist}_\rho(x, A) - d_\rho(x)/2 \geq d_\rho(x) \geq \mathrm{diam}_\rho(B).$$

The Basic Modulus Estimate (Fact 2.8) now provides the inequality

$$\mathrm{mod}_Q(A, B; \Omega) \leq c\left(\log\left(1 + L/d_\rho(x)\right)\right)^{1-Q}.$$

On the other hand, we have (see below) $\Sigma(A, B) \leq 8a^3H^2$, so by the Loewner condition,

$$q = \varphi(8a^3H^2) \leq \mathrm{mod}_Q(A, B; \Omega) \leq c\left(\log\left(1 + L/d_\rho(x)\right)\right)^{1-Q}.$$

Thus $\mathrm{dist}_\rho(x, A) \leq C d_\rho(x)$ where $C = C(m, a, H, M, Q, \varphi) = 1 + \exp\left((c/q)^{1/(Q-1)}\right)$.

It remains to check that $\Sigma(A, B) \leq 8a^3H^2$. We claim that $\mathrm{diam}(B) \geq d(x)/8aH^2$. We can assume that $B \subset B(x; d(x)/4a)$. Then by choosing $y \in B$ appropriately and appealing to Corollary 2.5 and Theorem E we find that

$$\rho(x)|x - y| \geq \frac{d_\rho(x, y)}{aH} \geq \frac{d_\rho(x)}{4aH} \geq \frac{\rho(x)d(x)}{8aH^2}.$$

Next we claim that $\mathrm{diam}(A) \geq d(x)/4a^2H^2$. Let u, v be the endpoints of A on $[w, \zeta)$ with w, u, x, ζ successive points along the geodesic. With Corollary 2.5 and Theorem E in mind, we see that $|u - x| \geq d(x)/2aH^2$, because otherwise we would have $d_\rho(x, u) \leq d_\rho(x)$ which contradicts $\mathrm{dist}_\rho(x, A) \geq 3d_\rho(x)/2$. Thus

$$a\,\mathrm{diam}(A) \geq a|u - v| \geq \ell[u, v] \geq |u - x| \geq d(x)/4aH^2.$$

Finally, $\min\{\mathrm{diam}(A), \mathrm{diam}(B)\} \geq d(x)/8a^2H^2$ by above, and

$$\mathrm{dist}(A, B) \leq \min\{|u - x|, |v - x|\} \leq \min\{\ell[u, x], \ell[v, x]\} \leq ad(x),$$

which gives $\Sigma(A, B) \leq 8a^3H^2$ as desired.

$\diamond$

7. Regular Spaces

We call (Ω, d, μ) *(Ahlfors-David) Q-regular* provided there is a constant M such that

$$\frac{1}{M} r^Q \leq \mu B(x; r) \leq M\, r^Q \qquad \text{for all } x \in \Omega \text{ and all } 0 < r \leq \mathrm{diam}\,\Omega.$$

Here the positive real number Q must equal the Hausdorff dimension of our space.

Thanks to Heinonen and Koskela [HK98, Theorem 3.6,(3.11)], we have the following information about Loewner control functions in a regular Loewner space.

7.1. Fact. *In a Q-regular Q-Loewner space, the optimal control function satisfies*

$$\varphi(t) \geq b/(\log t)^{Q-1} \qquad \textit{for all sufficiently large } t.$$

Here the constant b depends only on the associated data, and satisfies $b \leq c = 2^{Q+1}M/\log 2$.

The inequality between b and the 'spherical ring constant' c follows easily by considering appropriate subarcs of some curve joining two distinct points and using the Loewner condition in conjunction with the Spherical Ring Estimate (Fact 2.7). In particular, it is this fact which guarantees that $\alpha = (b/c)^{1/(Q-1)} \le 1$ in Theorem C.

In this same connection, we have a comment about the dimension of $\partial_\rho\Omega$. We saw in subsection 6.A that $\dim_{\mathcal{H}}(\partial_\rho\Omega) = \dim_{\mathcal{H}}(\partial\Omega)$. Of course there we used Theorem C to realize $\partial_\rho\Omega \subset \partial\Omega$, but more importantly, the Hausdorff measures are defined via the original distance on Ω. As pointed out in [BKR98, §7], it is more natural to analyze

$$\dim_\rho(\partial_\rho\Omega) = \dim_{\mathcal{H}}(\partial_\rho\Omega, d_\rho)$$

where we use Hausdorff measures defined via the distance d_ρ. Proceeding as in [BKR98, Theorem 7.1], the identity map $id : (\bar{\Omega}_\rho, d_\rho) \to (\Omega \cup \partial_\rho\Omega, d)$ is Hölder continuous with exponent α (by Theorem C), and therefore

$$\dim_\rho(\partial_\rho\Omega) \ge \alpha \dim(\partial_\rho\Omega) = \alpha \dim(\partial\Omega).$$

Here we assume that ρ is a conformal density on a bounded, uniform, regular, Loewner metric measure space.

Next we give pointwise estimates for a conformal density. See also Theorem 5.3.

7.2. Theorem. *Let ρ be a conformal density on a bounded, a-uniform, Q-regular, Q-Loewner metric measure space (Ω, d, μ). There is a constant C, depending only on the data associated with Ω, such that*

$$\frac{1}{C}\left(\frac{d(x)}{d(w)}\right)^{\beta-1} \le \frac{\rho(x)}{\rho(w)} \le C\left(\frac{d(w)}{d(x)}\right)^{\beta+1}$$

for all $x \in \Omega$. Here $w \in \Omega$ satisfies $d(w) = \sup_{x\in\Omega} d(x)$ (so, $1 \le \operatorname{diam}\Omega/d(w) \le 2a$), and $\beta = 1/\alpha \ge 1$ depends only on the data.

Proof. Fix $x \in \Omega$. The lower estimate is an easy consequence of Theorem C in conjunction with Corollary 2.5. Let E be an arc joining x to the sphere $S(x; d(x)/4a)$ inside the ball $B = B(x; d(x)/4a)$. Then

$$\frac{d(x)/4a}{\operatorname{diam}\Omega} \le \frac{\operatorname{diam} E}{\operatorname{diam}\Omega} \le \psi\left(\frac{\operatorname{diam}_\rho E}{\rho(w)d(w)}\right) \le C\left(\frac{H\rho(x)d(x)/2}{\rho(w)d(w)}\right)^\alpha.$$

To verify the upper inequality, we can assume that $d(x) < d(w)/2$ and $\rho(x)d(x) > 2H^2\rho(w)d(w)$, for otherwise the estimate is trivial. Let E, F respectively be arcs joining w, x to the spheres $S(w; d(w)/4a)$, $S(x; d(x)/4a)$ inside the balls $B(w; d(w)/4a)$, $B(x; d(x)/4a)$. Then $\Sigma(E, F) \le 8a^2 d(w)/d(x) = t$, so $\operatorname{mod}_Q(E, F; \Omega) \ge \varphi(t)$. On the other hand, each curve $\gamma \in \Delta(E, F; \Omega)$ has

$$\ell_\rho(\gamma) \ge \frac{\rho(x)d(x)}{4H} = L > \lambda = \frac{H}{2}\rho(x)d(x),$$

so the Basic Modulus Estimate gives

$$b\,(\log t)^{1-Q} \le \varphi(t) \le \operatorname{mod}_Q(E, F; \Omega) \le c\left(\log\left(1 + \frac{L}{\lambda}\right)\right)^{1-Q}.$$

Therefore

$$\left(8a^2\frac{d(w)}{d(x)}\right)^\beta = t^\beta \ge \frac{L}{\lambda} = \frac{1}{2H^2}\frac{\rho(x)d(x)}{\rho(w)d(w)}.$$

◇

We close this paper with the following observation. Suppose that ρ is a conformal density on a uniform locally Q-Loewner metric measure space (Ω, d, μ) which satisfies the Whitney ball modulus property. According to [BHK01, Theorem 6.39] and Theorem E, Ω_ρ is also locally Q-Loewner. It now follows that Ω_ρ is Q-regular: (VG) gives the upper mass condition while the lower mass condition is an easy consequence of the Loewner property. We conclude that Ω_ρ is a Q-regular locally Q-Loewner space which satisfies the Gehring-Hayman Inequality, the Separation Property, and has a Gromov hyperbolic quasihyperbolization (Ω, k_ρ).

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Department of Mathematics, University of Cincinnati, OH 45221
E-mail address: david.herron@math.uc.edu