QUASICONFORMAL DEFORMATIONS AND VOLUME GROWTH

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Abstract. We determine when a metric measure space admits a uniformizing density which has Ahlfors regular volume growth. A characterization of uniformizing conformal densities in terms of doubling and a lifting procedure are the key ingredients in our presentation. We also furnish an application characterizing metric doubling measures.

1. Introduction

This article is a companion paper to recent work ([BHK01], [BHR01], [BKR98]) of Bonk, Heinonen, Koskela and Rohde; we exploit the theory developed in [BHK01] to extend some results of [BHR01] to the setting of abstract metric measure spaces (Ω, d, μ). More importantly, we use our results to determine when such a space admits a quasiconformal deformation which is both uniformizing and such that the induced measure satisfies a natural volume growth estimate; roughly speaking, under certain reasonable minimal hypotheses, this occurs precisely when the conformal Assouad dimension of the space's Gromov boundary is small enough.

The reader is encouraged to consult Sections 2 and 4 which contain information including basic definitions, notation and terminology. We mention that by now uniform domains (regions in which points can be joined by twisted double cone arcs) are well recognized as being the 'nice' domains for quasiconformal function theory as well as many other areas of analysis (e.g., potential theory); see [Geh87] for Euclidean space and [CGN00] for the Carnot-Carathéodory setting. The uniformization theory developed by Bonk, Heinonen, and Koskela provides a two way correspondence between uniform spaces and Gromov hyperbolic spaces; we highly recommend a perusal of [BHK01].

In [BHR01] and [BKR98] the authors study conformal deformations of the unit ball in Euclidean space. The primary object of our study here is the geometry of quasiconformal deformations of an abstract metric measure space (Ω, d, μ); however, our results are new even for domains in Euclidean space. Following BKHR, we consider a metric-density ρ on Ω and $\Omega_\rho = (\Omega, d_\rho, \mu_\rho)$ denotes the deformed space (see subsection 2.C). We are interested in the situation when this new space Ω_ρ is uniform (see subsection 2.E); we describe this by calling such ρ a *uniformizing density*. According to [BHK01, Props.2.8,4.5]: Every proper geodesic Gromov hyperbolic space can be uniformized, and, there is a natural canonical proper geodesic space associated with any abstract domain, namely, its quasihyperbolization (see subsections 2.D-2.F, 4.B). However, in general the associated measure (see (2.1)) may fail to

Date: May 18, 2005.

1991 *Mathematics Subject Classification.* Primary: 30C65; Secondary: 30C20.

Key words and phrases. uniform space, conformal density, doubling, metric doubling measure, Assouad dimension, Gromov hyperbolic.

The author is grateful for the hospitality he received at both the University of Michigan and the University of Jyväskylä where much of this research was done. The author was partially supported by the Charles Phelps Taft Research Center.

have Ahlfors regular volume growth; e.g., if we start with the unit ball in Euclidean space, quasihyperbolize and then BHK-uniformize, our new measure will have exponential volume growth. Here we address the problem of deciding when there exists a uniformizing density which is also conformal; in particular, the associated measure (2.1) should have Ahlfors regular volume growth. Our main result provides the following answer to this problem.

A. **Theorem.** *Let Ω be an abstract domain with bounded Q-geometry. Suppose Ω admits a bounded uniformizing conformal density. Then Ω has a Gromov hyperbolic roughly starlike quasihyperbolization and the conformal Assouad dimension of its Gromov boundary is strictly less than Q. The converse holds too, provided the Gromov boundary of Ω is uniformly perfect.*

The above result is quantitative: the asserted constants depend only on the data associated with Ω and the density.

We consider metric measure spaces (Ω, d, μ) which are abstract domains having bounded Q-geometry. According to [BHK01, Thm.3.6]), such a space can be boundedly conformally uniformized only if it has a Gromov hyperbolic roughly starlike quasihyperbolization. Equipped with this information, we consider metric measure spaces (Ω, d, μ) which satisfy the following *basic minimal hypotheses*:

Ω is an abstract domain having bounded Q-geometry and a
Gromov hyperbolic roughly starlike quasihyperbolization.

Precise definitions are stated in subsections 2.B, 2.D, 3.A, 4.B; roughly, these hypotheses ensure that Ω has 'enough' of the local properties enjoyed by domains in Euclidean space. The *data* associated with these basic hypotheses consists of six parameters: Q (the 'dimension'), M, m, λ (the bounded geometry constants), Δ (the Gromov hyperbolicity constant) and κ (the rough starlike constant). As examples we mention that complete simply connected Riemannian manifolds whose sectional curvatures are everywhere less than a negative constant (e.g., all CAT(-1) spaces) are Gromov hyperbolic and of bounded geometry if the curvatures are pinched. For further examples, see subsections 4.B, 3.A and especially [BHK01] and the references therein. Also, see [BB03, Thm.3.1, Prop.4.19] for useful criteria for determining when a quasihyperbolization is roughly starlike, and [BS00, §§7-8] for a method to construct a roughly starlike Gromov hyperbolic space with a given Gromov boundary.

Notice that Theorem A provides three increasingly more stringent obstructions for finding a conformal uniformizing density. For example, the BHK-uniformization of n-dimensional complex hyperbolic space is an abstract domain, with bounded $2n$-geometry, which cannot be quasiconformally uniformized (when $n > 1$) because its Gromov boundary is the n-dimensional compactified Heisenberg group which has conformal Assouad dimension $2n$. In a different direction: Every Gromov hyperbolic plane domain can be uniformized via a conformal mapping [BHK01, Thm.1.12], and therefore its Gromov boundary has conformal Assouad dimension strictly less than 2.

The Bonk-Heinonen-Koskela uniformization theory is a crucial tool employed in all our arguments and permits us to replace the space Ω with a bounded uniform space Ω_ε where the geometry is more transparent; see subsection 2.F and the introduction to Section 3. A key ingredient in our proof of Theorem A is the following generalization of [BHR01, Prop.2.11]. In particular, it asserts that a conformal density on a bounded uniform space is uniformizing if and only if the associated measure (2.1) is doubling. (See subsection 3.B for the precise definition of a doubling conformal density.)

B**. Theorem.** *Assume the basic minimal hypotheses. Let Ω_ε be any BHK-uniformization of Ω. Suppose ρ is a conformal density on Ω. Then the following are quantitatively equivalent:*

(a) *ρ is doubling on Ω.*
(b) *Ω_ρ is bounded and uniform.*
(c) *Ω_ρ is bounded and Q-Loewner.*
(d) *Ω_ρ is bounded, Q-Loewner and Ahlfors Q-regular.*
(e) *the identity map $\Omega_\rho \to \Omega_\varepsilon$ is quasisymmetric.*

Again, this result is quantitative: the asserted constants depend only on the data associated with Ω and ρ, and the related data. Also, we point out that the proof of (b) shows that the quasihyperbolic geodesics in Ω will be uniform arcs in Ω_ρ.

Next we communicate the primary tool employed in our proof of Theorem A. It is based on a lifting procedure discussed in [BKR98, 2.7] and established for the Euclidean unit ball as [BHR01, Prop.1.25]. See (3.9), (3.15) and (4.8) for the definitions of ρ_ν (the *lift* of ν) and $\delta_{\nu,1/P}$ (the quasimetric determined by ν).

C**. Theorem.** *Assume the basic minimal hypotheses, that the Gromov boundary of Ω is uniformly perfect, and that $P < Q$. Suppose ν is a P-dimensional metric doubling measure on $\partial_G\Omega$. Then the lift $\rho = \rho_\nu$ of ν is a doubling conformal density on Ω and the natural map $(\partial_\rho\Omega, d_\rho) \to (\partial_G\Omega, \delta_{\nu,1/P})$ is bilipschitz.*

Another application of Theorems B and C provides a characterization of doubling conformal densities in terms of metric doubling measures. We say that a metric-density ρ on Ω is *induced by a metric doubling measure* ν on $\partial_G\Omega$ if there exists a constant $C \geq 1$ such that

$$C^{-1}\rho(x) \leq \rho_\nu(x) \leq C\rho(x) \qquad \text{for } \mu\text{-a.e. } x \in \Omega.$$

D**. Theorem.** *Assume the basic minimal hypotheses, that the Gromov boundary of Ω is uniformly perfect, and $P < Q$. A metric-density ρ on Ω is a doubling conformal density, with $(\partial_\rho\Omega, d_\rho)$ Ahlfors P-regular, if and only if ρ is induced by some P-dimensional metric doubling measure on $\partial_G\Omega$.*

We gratefully acknowledge helpful discussions with Mario Bonk, and are indebted to Juha Heinonen for suggesting this line of inquiry and especially for hours of blackboard sessions in his office and numerous conversations on the phone, in the sauna, and along ski trails. We also thank the referee for their careful reading of this paper.

2. Preliminaries

2.A. **General Information.** This paper is organized as follows: The present section contains information about abstract domains, quasiconformal deformations, quasihyperbolic distance, uniform spaces and uniformization. Basic information (definitions, terminology, and facts) regarding general metric spaces (e.g., Gromov hyperbolicity and doubling spaces) can be found in Section 4. In particular, subsection §4.F has a table listing most of our definitions and where they can be found in this article. The proofs of our four main theorems are presented in Section 3 where we also discuss bounded geometry and doubling conformal densities and present a local doubling property of quasihyperbolic space.

Our notation is relatively standard and, for the most part, conforms with that of [BHK01]. We write $C = C(a, \ldots)$ to indicate a constant C which depends only on the parameters $a, \ldots$; the notation $A \lesssim B$ means there exists a finite constant c with $A \leq cB$, and $A \simeq B$ means that both $A \lesssim B$ and $B \lesssim A$ hold. Typically $a, b, c, C, K, \ldots$ will be constants that depend on various parameters, and we try to make this as clear as possible often giving explicit values, however, at times C will denote some constant whose value depends only on the data present but may differ even on the same line of inequalities.

2.B. **Abstract Domains.** We work in the setting of a metric measure space (Ω, d, μ) with Ω a locally compact, rectifiably connected, non-complete metric space and μ a Borel regular measure with dense support. We indicate this by saying that Ω is an *abstract domain*; an important example of such a space is, of course, a proper subdomain of Euclidean space with either Euclidean distance or the induced Euclidean length distance. We let $\bar{\Omega}$ denote the metric completion of Ω, and we call $\partial\Omega = \bar{\Omega} \setminus \Omega$ the metric boundary of Ω. Then $d(x) = \operatorname{dist}(x, \partial\Omega)$ is the distance from a point $x \in \Omega$ to the boundary of Ω; note that as $\partial\Omega$ is closed in $\bar{\Omega}$, we have $d(x) > 0$ for all $x \in \Omega$. We let $|x - y| = d(x, y)$ denote the distance between points x, y and $B(x; r) = \{y : |x - y| < r\}$ ($S(x; r) = \{y : |x - y| = r\}$) is the open ball (sphere) of radius r centered at the point x. When $B = B(x; r)$, $cB = B(x; cr)$. We make extensive use of the notation

$$B(x) := B(x; d(x)).$$

Unless explicitly indicated otherwise, the adjective *locally* means that the modified property or condition holds in all Whitney-type balls $\lambda B(x)$ where $0 < \lambda < 1$ is some fixed constant which we call the *Whitney ball constant*; when there are several such local conditions in play, we take λ to be the minimum of all the associated Whitney ball constants. For example, we declare Ω to be *locally quasiconvex* if there exists a constant $a \geq 1$ such that for all $z \in \Omega$, points $x, y \in \lambda B(z)$ can be joined by a rectifiable arc α in Ω with $\ell(\alpha) \leq a|x - y|$; we abbreviate this by the phrase 'Ω is locally a-quasiconvex' (again, here it is understood that there is some Whitney ball constant $0 < \lambda < 1$, which may also depend on other parameters).

There is a second, intrinsic, distance function on Ω, the so-called *length distance* given by

$$\ell(x, y) = \inf\{\ell(\gamma) : \gamma \text{ a rectifiable curve joining } x, y \text{ in } \Omega\}.$$

Since $|x - y| \leq \ell(x, y)$ for all x, y, the identity map $(\Omega, \ell) \to (\Omega, d)$ is Lipschitz continuous. It is important to know when this map will be a homeomorphism (cf. [BHK01, Lma.A.4,p.92]). Notice that the identity map $(\Omega, d) \to (\Omega, \ell)$ is uniformly locally Lipschitz when Ω is locally quasiconvex.

2.C. **Quasiconformal Deformations.** Given a positive Borel measurable function ρ on Ω, we let

$$\ell_\rho(\gamma) = \int_\gamma \rho \, ds$$

denote the ρ-length of a rectifiable curve γ, and we define the ρ-distance between two points x, y by

$$d_\rho(x, y) = \inf\{\ell_\rho(\gamma) : \gamma \text{ a rectifiable curve joining } x, y \text{ in } \Omega\}.$$

The careful reader no doubt recognizes that, in general, $d_\rho(x, y)$ could be zero or even infinite; in order to ensure that d_ρ be an honest distance function, we must require that $0 < d_\rho(x, y) < \infty$ for all points $x, y \in \Omega$. We designate this by calling such ρ a Borel *metric-density* on Ω. One way to guarantee this is to ask that ρ be locally bounded away from zero and infinity. In practice, our densities will always satisfy a Harnack inequality—see below—and so this is never a problem for us.

The ρ-balls (etc.) are written as $B_\rho(x;r)$; these are the metric balls in Ω_ρ, so $B_\rho(x;r) = \{y \in \Omega : d_\rho(x,y) < r\}$. We also form a new measure μ_ρ given by

$$\mu_\rho(E) = \int_E \rho^Q \, d\mu. \tag{2.1}$$

Here Q is generally understood from context, and is usually the Hausdorff dimension of Ω. Thus each metric-density ρ gives rise to a new metric measure space $\Omega_\rho = (\Omega, d_\rho, \mu_\rho)$ which is a deformation of Ω. (In subsection 4.E we examine another method for deforming the geometry of Ω which is based on the measure.) We declare ρ to be *bounded* if the deformed space Ω_ρ is bounded, i.e., if $\operatorname{diam}_\rho(\Omega) < \infty$.

When Ω_ρ is non-complete (which will typically be the case for us), we can form $\partial_\rho\Omega = \bar{\Omega}_\rho \setminus \Omega_\rho$ and define $d_\rho(x) = \operatorname{dist}_\rho(x, \partial_\rho\Omega)$. In this setting we also employ the notation $B_\rho(x) = B_\rho(x; d_\rho(x))$; thus $\lambda B_\rho(x)$ is a Whitney ball in Ω_ρ.

We are especially interested in the metric-densities ρ for which Ω_ρ is a uniform space, and we call such a ρ a *uniformizing density* (which implicitly includes the hypothesis that Ω_ρ is non-complete). The Bonk-Heinonen-Koskela theory produces uniformizing densities on proper geodesic Gromov hyperbolic spaces; see subsections 2.E,2.F,4.B. Some other classes of metric-densities which we wish to single out for attention include the following: Harnack, Ahlfors, Koebe, conformal; their definitions follow. We let H_ρ, A_ρ, K_ρ denote the parameters associated with these densities. (See subsection 3.B for the definition and properties of doubling densities.)

We call ρ a *Harnack density* provided it satisfies a uniform local Harnack type inequality: for all points x in Ω,

$$\frac{1}{H} \le \frac{\rho(y)}{\rho(x)} \le H \qquad \text{for all } y \in \lambda B(x). \tag{H}$$

Here $H = H_\rho \ge 1$ and $0 < \lambda < 1$ (generally λ will be small). Note that in contrast to the situation in [BKR98, p.637], the validity of (H) for some $0 < \lambda < 1$ need not mean a similar set of inequalities will hold for $\lambda = 1/2$. The condition (H) provides local control and permits the use of standard chaining type arguments; e.g., see Lemma 2.8.

We call ρ an *Ahlfors density* if the associated metric measure space $\Omega_\rho = (\Omega, d_\rho, \mu_\rho)$ is Ahlfors *upper* Q-regular (cf. §4.D); i.e., if μ_ρ satisfies a global upper Ahlfors Q-regular volume growth estimate: there is a constant $A = A_\rho$ such that for all points x in Ω,

$$\mu_\rho[B_\rho(x;r)] \le A\, r^Q \qquad \text{for all } r > 0. \tag{A}$$

The positive real number Q is generally the Hausdorff dimension of our space; it must agree with the number Q appearing in the definition of a Loewner space (a notion discussed in subsection 4.D). The volume growth condition (A) ensures that Ω_ρ satisfies an upper mass condition and so provides modulus estimates via Facts 4.3 and 4.4.

Below we discuss the density $1/d$ which determines the quasihyperbolic distance; of course this is a continuous Harnack density, but in general $1/d$ does not satisfy the volume growth requirement (A).

We call ρ a *Koebe density* if Ω_ρ is non-complete and there is a constant $K = K_\rho$ such that $d_\rho(x) = \operatorname{dist}_\rho(x, \partial_\rho\Omega)$ enjoys the property

$$\frac{1}{K} \le \frac{d_\rho(x)}{\rho(x) d(x)} \le K \qquad \text{for all } x \in \Omega. \tag{K}$$

(Note that when ρ is a Harnack density, $d_\rho(x) \geq (\lambda/H)\rho(x)d(x)$ always holds, and so it is the upper estimate which is needed.) For example, if $\rho = |f'|$ where f is a holomorphic homeomorphism defined in a subdomain Ω of the complex plane, then a classical theorem in univalent function theory asserts that ρ is a Koebe density with constant $K = 4$. As another example we mention that any Harnack Ahlfors density on a uniform Loewner space (with sufficiently 'thick' boundary) is a Koebe density; see [Her04, Thm.E]. We point out that when ρ is a Koebe density on (Ω, d), the identity map $(\Omega, k) \to (\Omega_\rho, k_\rho)$ is easily seen to be K_ρ-bilipschitz; here (Ω_ρ, k_ρ) denotes the quasihyperbolization of Ω_ρ.

We employ the terminology *conformal density* for a metric-density which is Harnack, Ahlfors, and Koebe. Note that our terminology does not agree exactly with that of [BKR98], [BHR01], nor [Her04]; in both the former and latter of these it is a theorem that Harnack Ahlfors densities are Koebe (hence conformal) under the appropriate hypotheses on Ω ([BKR98, Prop.6.2], [Her04, Thm.E]). A basic example of a conformal density is $\rho = |f'|$ for any holomorphic homeomorphism $|f'|$ defined in a subdomain of the complex plane; we refer to [BKR98, Section 2] for other examples of conformal densities on the Euclidean unit ball.

We pause to make a few comments concerning our terminology. The reader no doubt has encountered deformations of Euclidean domains $\Omega \subset \mathbf{R}^n$ by continuous densities ρ; in this setting Ω_ρ is a conformal deformation of Ω, meaning that the identify map $\Omega \to \Omega_\rho$ is conformal. However, in our more general setting, even for the case $\rho = 1$ say, the identity map $\Omega \to \Omega_\rho = \Omega_\ell$ may fail to be quasiconformal (e.g., if Ω does not satisfy some sort of local quasiconvexity condition). A similar phenomenon holds for Borel metric-densities, even for domains $\Omega \subset \mathbf{R}^n$. Nonetheless, when Ω is locally quasiconvex and ρ is a Harnack metric-density, then Lemma 2.2 below reveals that Ω_ρ is a quasiconformal deformation of Ω (and according to Proposition 2.4 even a quasisymmetric deformation under the right circumstances). Thus the phrase 'ρ is a conformal density' does not necessarily mean that the identity map $\Omega \to \Omega_\rho$ is quasiconformal (unless Ω is locally quasiconvex).

Here are some especially useful estimates which also provide information concerning the identity map $\Omega \to \Omega_\rho$ for certain densities. Roughly speaking, this map is locally bilipschitz (therefore quasiconformal) for Harnack densities and uniformly locally quasisymmetric for Harnack Koebe densities, provided Ω is locally quasiconvex. Proposition 2.4 gives a significant strengthening of this result.

2.2. **Lemma.** *Let $\rho : \Omega \to (0, \infty)$ be a Harnack density on a locally a-quasiconvex abstract domain (Ω, d). Put $\eta = \lambda/2a$. Then for all $z \in \Omega$,*

$$\frac{1}{H} \leq \frac{d_\rho(x, y)}{\rho(z)|x - y|} \leq aH \qquad \text{for all points } x \neq y \text{ in } \eta B(z);$$

in particular,

$$\frac{1}{H} \leq \frac{\operatorname{diam}_\rho[\eta B(z)]}{\eta\rho(z)d(z)} \leq 2aH.$$

If ρ is also a Koebe density, then for all $0 \leq \vartheta \leq \lambda/2C$ and all $x \in \Omega$,

$$C^{-1}\vartheta B(x) \subset \vartheta B_\rho(x) \subset C\vartheta B(x),$$

where $C = aHK$. Here $H = H_\rho$, $K = K_\rho$ and λ is the Whitney ball constant.

Proof. Fix $z \in \Omega$, put $d = d(z)$, and let $x, y \in \eta B(z)$. To see the lower bound, let γ be any rectifiable arc which joins x, y. If $\gamma \subset \lambda B(z)$, then by (H) $\ell_\rho(\gamma) \geq H^{-1}\rho(z)\ell(\gamma) \geq$

$H^{-1}\rho(z)|x-y|$. If γ leaves $\lambda B(z)$, then γ contains subarcs $\alpha, \beta \subset \gamma \cap \lambda B(z)$ which join x, y (respectively) to the sphere $S(z; \lambda d)$. Thus

$$\ell_\rho(\gamma) \geq \ell_\rho(\alpha) + \ell_\rho(\beta) \geq 2H^{-1}\rho(z)(\lambda - \eta)d \geq H^{-1}\rho(z)|x-y|.$$

For the opposite inequality, let α be the promised arc joining x, y with $\ell(\alpha) \leq a|x-y| \leq 2a\eta d$. Then $\alpha \subset \lambda B(z)$ (for if $w \in \alpha$, then $|z-w| \leq \min\{|z-x|+\ell(\alpha(x,w)), |z-y|+\ell(\alpha(y,w))\} < \eta d + \ell(\alpha)/2 \leq (1+a)\eta d \leq \lambda d$). Appealing to (H) we obtain

$$d_\rho(x,y) \leq \ell_\rho(\alpha) \leq H\,\rho(z)\ell(\alpha) \leq aH\,\rho(z)|x-y|.$$

Now assume that ρ is also a Koebe density and fix $\eta = \lambda/2a$, $0 < \vartheta \leq \lambda/2C$ and $x \in \Omega$. The rightmost containment follows from the first part because $\vartheta B_\rho(x) \subset \eta B(x)$; for then $d_\rho(x,y) < \vartheta d_\rho(x)$ implies

$$H^{-1}\rho(x)|x-y| \leq d_\rho(x,y) < \vartheta d_\rho(x) \leq \vartheta K\rho(x)d(x) \quad \text{so} \quad |x-y| < C\vartheta d(x).$$

To see the above, suppose there were a point $y \in \vartheta B_\rho(x) \setminus \eta B(x)$. Then any arc γ joining x, y would contain a subarc $\alpha \subset \eta B(x)$ with $\ell(\alpha) \geq \eta d(x)$, so we would have $\ell_\rho(\gamma) \geq \ell_\rho(\alpha) \geq H^{-1}\rho(x)\ell(\alpha) \geq (\eta/H)\rho(x)d(x)$, which would yield

$$(\eta/H)\rho(x)d(x) \leq d_\rho(x,y) < \vartheta d_\rho(x) \leq \vartheta K\rho(x)d(x) \quad \text{giving} \quad \vartheta > \eta(HK)^{-1} = \frac{\lambda}{2C}.$$

It remains to verify the leftmost asserted containment. This also follows from the first part, because $\vartheta/C \leq \eta$ gives us $|x-y| < (\vartheta/C)d(x)$ implying $d_\rho(x,y) \leq aH\rho(x)|x-y| < (aH\vartheta/C)\rho(x)d(x) \leq \vartheta d_\rho(x)$. □

Using the above, we easily deduce that Harnack Koebe densities on locally Ahlfors regular abstract domains are locally Ahlfors regular.

2.3. **Corollary.** *Let ρ be a Harnack density on a locally a-quasiconvex abstract domain (Ω, d, μ). Suppose that μ is locally upper Ahlfors Q-regular (or, locally lower Ahlfors Q-regular, resp.). Then for all $x \in \Omega$ and all $0 \leq r \leq (\lambda/2a^2H)\rho(x)d(x)$,*

$$\mu_\rho[B_\rho(x;r)] \leq M(H^2r)^Q \quad \textit{(or, } \mu_\rho[B_\rho(x;r)] \geq M^{-1}(r/aH^2)^Q, \textit{ resp.).}$$

Here $H = H_\rho$, $M = M_\mu$ and λ is the Whitney ball constant.

Proof. It is not hard to see that both balls $B(x; r/aH\rho(x))$ and $B_\rho(x;r)$ lie in $\eta B(x)$ where $\eta = \lambda/2a$, so Lemma 2.2 corroborates that

$$B(x; r/aH\rho(x)) \subset B_\rho(x;r) \subset B(x; Hr/\rho(x)) \subset \lambda B(x).$$

The mass estimates then follow from the local Q-regularity of μ. □

Another immediate consequence of Lemma 2.2 is that the identity map $\Omega \to \Omega_\rho$ is metrically quasiconformal with linear dilatation aH^2. In addition, because of the definition of the associated measure (see (2.1)), a straightforward calculation reveals that this identity map is geometrically quasiconformal with inner dilatation H^Q and outer dilatation $(aH)^Q$. (Here we assume a Harnack density on a locally quasiconvex Ω.) It is therefore natural to inquire about possible quasisymmetry properties of this identity map.

Heinonen and Koskela proved that a quasiconformal map of bounded Ahlfors regular spaces, with domain a Loewner space and target space linearly locally connected, is in fact quasisymmetric [HK98, Thm.4.9]. We present the following analog of their result; note that our domain space is *not* assumed to be Ahlfors regular. Recall that a space is *c-linearly* *locally connected* provided points in a ball $B(x;R)$ (respectively, not in $\bar{B}(x;R)$) can be joined by a continuum in $B(x;cR)$ (respectively, in the complement of $\bar{B}(x;R/c)$). Both uniform spaces as well as Ahlfors regular Loewner spaces enjoy this property; see [HK98, Thm.3.13].

2.4. **Proposition.** *Let Ω be a bounded locally quasiconvex Q-Loewner space. Suppose ρ is a conformal density on Ω with Ω_ρ a bounded linearly locally connected space. Then the identity map $\Omega \to \Omega_\rho$ satisfies the weak-quasisymmetry condition*

$$\forall x,y,z \in \Omega : \quad |x-y| \le |x-z| \implies d_\rho(x,y) \le L d_\rho(x,z)$$

for some constant L which depends only on the data associated with Ω, ρ, Ω_ρ, and the ratios r, q given in the proof.

Proof. We are assuming that Ω is a bounded locally a-quasiconvex abstract domain and ρ is a Harnack Ahlfors Koebe metric-density on Ω with associated constants $H = H_\rho$, $A = A_\rho$, $K = K_\rho$. According to Lemma 2.2, the desired inequality holds with $L = aH^2$ for points x, y, z in any Whitney ball $(\lambda/2a)B(w)$.

Put $\sigma = \lambda/8a$ and $\tau = \sigma/2a = \lambda/16a^2$. Thus we have weak-quasisymmetry in each Whitney ball $4\sigma B(w)$. Fix a point $w \in \Omega$ and set

$$r = \frac{\operatorname{diam}(\Omega)}{d(w)} \quad , \quad q = \frac{\operatorname{diam}_\rho(\Omega)}{d_\rho(w)}.$$

Let $x, y, z \in \Omega$ and suppose $|x-y| \le |x-z|$. Assume that $d_\rho(x,y) > L d_\rho(x,z)$. We show that L cannot be too large. Notice that as $d_\rho(x,y) \le \operatorname{diam}_\rho(\Omega)$, we also have $d_\rho(x,z) \le (q/L) d_\rho(w)$. We consider the alternatives $x \in \sigma B(w)$ or $x \notin \sigma B(w)$, and subcases of these.

Suppose first that $x \in \sigma B(w)$. If $z \in 2\sigma B(w)$, then x, y, z all belong to $4\sigma B(w)$ and so $L \le aH^2$. Assume $z \notin 2\sigma B(w)$. Then $d_\rho(x,z) \ge (\rho(w)/H)\sigma d(w) \ge (\sigma/HK) d_\rho(w)$, so $L \le qHK/\sigma$.

Now suppose that $x \notin \sigma B(w)$. Then $d_\rho(x,w) \ge (\sigma/HK) d_\rho(w)$, so from above we deduce $d_\rho(x,z) \le (qHK/\sigma L) d_\rho(x,w)$. Writing $R = d_\rho(x,z)$ we thus have $d_\rho(x,y) \ge LR$ and $d_\rho(x,w) \ge \varepsilon LR$ with $\varepsilon = \sigma/qHK$. Since Ω_ρ is c-linearly locally connected, there is a continuum E joining x, z in $B_\rho(x;cR)$.

Assume $y \notin \tau B(w)$. Appealing to linear local connectivity again, we obtain a continuum F joining y, w in $\Omega_\rho \setminus B_\rho(x; \varepsilon LR/c)$. Since Ω_ρ is Ahlfors upper Q-regular with constant A, we can use the spherical ring estimate Fact 4.4 to assert that

$$\operatorname{mod}_Q(E, F; \Omega_\rho) \le C(\log[\varepsilon c^{-2} L])^{1-Q},$$

where $C = 2^{Q+1} A/\log 2$. Looking back in Ω we see that $\operatorname{dist}(E,F) \le |x-y| \le |x-z| \le \operatorname{diam}(E)$ and $\operatorname{diam}(F) \ge |y-w| \ge \tau d(w) \ge (\tau/r)|x-y|$, so

$$\Sigma(E,F) = \operatorname{dist}(E,F)/\min\{\operatorname{diam}(E), \operatorname{diam}(F)\} \le r/\tau.$$

Since Ω is Q-Loewner (see §4.D), say with control function φ, we deduce that

$$\varphi(r/\tau) \le \operatorname{mod}_Q(E,F;\Omega) \le (aH)^Q \operatorname{mod}_Q(E,F;\Omega_\rho) \le (aH)^Q C(\log[\varepsilon c^{-2} L])^{1-Q}.$$

This provides the estimate $L \le \sigma^{-1} q c^2 HK \exp[((aH)^Q C/\varphi(r/\tau))^{1/(Q-1)}]$.

Finally, assume $y \in \tau B(w)$. Since Ω is locally a-quasiconvex and $\tau < \lambda$, there is an arc α joining y, w with $\ell(\alpha) \le a|y-w| \le (\sigma/2)d(w)$. Thus there is a continuum D joining y, w with $D \subset (\sigma/2)B(w)$ and $\operatorname{diam}(D) \ge \tau d(w)$; if necessary, we add to α a subarc joining w to $\partial[\tau B(w)]$. Since $x \notin \sigma B(w)$, we find that $\operatorname{dist}_\rho(x, D) \ge (\sigma/2HK) d_\rho(w)$ and so $D \subset \Omega_\rho \setminus B_\rho(x;(\sigma/2HK)d_\rho(w))$. Appealing once again to the spherical ring estimate Fact 4.4 we obtain

$$\mathrm{mod}_Q(E,D;\Omega_\rho) \le C(\log[\sigma(2qcHK)^{-1}L])^{1-Q}.$$

Back in Ω we have $\mathrm{dist}(E,D) \le |x-y| \le |x-z| \le \mathrm{diam}(E)$ and $\mathrm{diam}(D) \ge \tau d(w) \ge (\tau/r)|x-y|$, so again $\Sigma(E,D) \le r/\tau$ and we conclude that

$$\varphi(r/\tau) \le \mathrm{mod}_Q(E,D;\Omega) \le (aH)^Q \,\mathrm{mod}_Q(E,D;\Omega_\rho) \le (aH)^Q C(\log[\sigma(2qcHK)^{-1}L])^{1-Q}.$$

This provides the estimate $L \le \sigma^{-1}2qcHK\exp[((aH)^Q C/\varphi(r/\tau))^{1/(Q-1)}]$. □

2.5**. Corollary.** *Let Ω be a bounded quasiconvex Q-Loewner space. Suppose ρ is a conformal density on Ω and Ω_ρ is a bounded Q-Loewner space. Then the identity map $\Omega \to \Omega_\rho$ is quasisymmetric with a distortion function which depends only on the data associated with Ω, ρ, Ω_ρ, and the ratios r, q given in the proof of Proposition 2.4.*

Proof. Since Ω_ρ is Q-Loewner and upper Ahlfors Q-regular, it is an Ahlfors Q-regular linearly locally connected quasiconvex space; see [HK98, Thms.3.6,3.13] (and also the proof of Lemma 3.1). Therefore, by work of Tukia and Väisälä [TV80, Thm.2.15], the desired quasisymmetry is equivalent to weak-quasisymmetry in our setting. (Note that quasiconvexity implies pseudoconvexity and Ahlfors regularity implies doubling.) □

Now we come to a simple idea which nonetheless plays a crucial role in the proofs of our main theorems. Suppose τ is a metric-density on Ω and σ a density on $\Delta = \Omega_\tau$. We wish to pull-back σ via the formula $\rho = \sigma\,\tau$ to obtain a metric-density ρ on Ω. This is fine, assuming that Ω and Δ are homeomorphic, but we of course also want ρ to enjoy the same properties that σ does. It is straightforward to see that $\Omega_\rho = \Delta_\sigma$ (i.e., if $(\Delta,\delta,\nu) = \Delta = \Omega_\tau = (\Omega, d_\tau, \mu_\tau)$, then $d_\rho = \delta_\sigma$ and $\mu_\rho = \nu_\tau$); thus ρ and σ are simultaneously Ahlfors densities (or not) with the same constant. It is also not difficult to see that ρ will be Koebe (with $K_\rho = K_\sigma K_\tau$) whenever both σ and τ are Koebe densities. However, the Harnack condition is a bit more subtle; the main point is that we need each Whitney ball $\lambda B(x)$ in Ω to lie inside a Whitney ball $\kappa B_\tau(x)$ in Δ. It is not difficult to obtain this condition from Lemma 2.2, provided Ω is locally quasiconvex.

We also consider the situation when τ and ρ are given metric-densities on Ω and we wish to push-forward ρ to a density σ (via the formula $\sigma = \rho\,\tau^{-1}$) on $\Delta = \Omega_\tau$. Again, σ will be a metric-density (assuming Ω and Δ are homeomorphic and τ is locally bounded above), ρ and σ will simultaneously be (or not be) Ahlfors densities, and σ will be Koebe whenever both ρ and τ are Koebe densities. Also, a Harnack condition will follow for σ if both τ and ρ are Harnack and each Whitney ball $\kappa B_\tau(x)$ in Δ lies inside a Whitney ball $\lambda B(x)$ in Ω. Here though, because the τ^{-1} deformation of Δ is Ω_ℓ rather than Ω, this condition need not be true. (It is easy to construct locally quasiconvex Ω so that $\rho(x) = d(x)$ fails to be Harnack on Ω_ℓ.) However, if τ is also known to be Koebe, then we can appeal to Lemma 2.2.

We utilize these ideas extensively, and so record the following facts. See also Corollary 2.18.

2.6. **Lemma.** *Suppose (Ω,d,μ) is a locally a-quasiconvex abstract domain. Let τ be a positive Borel function on Ω which is locally bounded away from 0 and ∞. Put $\Delta = \Omega_\tau$. If σ is a metric-density on Δ, then its pull-back $\rho = \sigma\,\tau$ is a metric-density on Ω, $\Omega_\rho = \Delta_\sigma$, and*

(a) *ρ and σ either are, or are not, both Ahlfors regular (with $A_\rho = A_\sigma$),*
(b1) *if σ,τ are both Koebe, then so is ρ with $K_\rho = K_\sigma K_\tau$,*
(c1) *if σ,τ are both Harnack, then so is ρ with $H_\rho = H_\sigma H_\tau$.*

On the other hand, if ρ is a metric-density on Ω, then its push-forward $\sigma = \rho\tau^{-1}$ is a metric-density on Δ, $\Delta_\sigma = \Omega_\rho$, (a) *holds, and*

(b2) *if ρ, τ are Koebe, then so is σ with $K_\sigma = K_\rho K_\tau$,*

(c2) *if ρ, τ are Harnack and τ is Koebe, then σ is Harnack with $H_\sigma = H_\rho H_\tau$.*

Proof. To see that (c1) holds: The Harnack inequalities for σ and τ yield the same for ρ provided we can produce a Whitney ball constant η such that $\eta B(x) \subset \kappa B_\tau(x)$, where κ is the Whitney ball constant for σ. We start by putting $H = H_\tau$, letting λ be the Whitney ball constant for τ and the local quasiconvexity, and setting $\eta = \min\{\lambda/2a, \kappa\lambda/aH^2\}$. Since τ is Harnack, $d_\tau(x) \ge (\lambda/H)\tau(x)d(x)$. Therefore, as Ω is locally quasiconvex, we can appeal to Lemma 2.2 to see that for all $y \in \eta B(x)$,

$$d_\tau(x,y) \le a\eta H\tau(x)d(x) \le (a\eta H^2/\lambda)d_\tau(x) \le \kappa d_\tau(x);$$

thus $\eta B(x) \subset \kappa B_\tau(x)$.

To see that (c2) holds: Here we use the last part of Lemma 2.2 taking $\vartheta = \lambda/(2aHK)$ where λ is the minimum of all Whitney ball constants and $H = H_\tau$, $K = K_\tau$. Then $C\vartheta = (aHK)\vartheta = \lambda/2$, so $\vartheta B_\tau(x) \subset \lambda B(x)$ and the Harnack inequalities for ρ and τ will yield the same for σ. $\square$

2.D. **Quasihyperbolic Distance and Geodesics.** The quasihyperbolic distance is defined by

$$k(x,y) = k_\Omega(x,y) = \inf \ell_k(\gamma) = \inf \int_\gamma \frac{ds}{d(z)}$$

where the infimum is taken over all rectifiable curves γ which join x, y. The *quasihyperbolization* of an abstract domain (Ω, d) is the metric space (Ω, k) obtained by using quasihyperbolic distance; thanks to the Hopf-Rinow theorem, it is always complete, proper and geodesic provided the identity map $(\Omega, \ell) \to (\Omega, d)$ is a homeomorphism; see [BHK01, Prop.2.8], [Gro99, p.9], [BBI01, p.51], [BH99, p.35].

We call the geodesics in (Ω, k) *quasihyperbolic geodesics*; see subsection 4.A. Note that when ρ is a Koebe density on Ω, the identity map $(\Omega, k) \to (\Omega_\rho, k_\rho)$ is bilipschitz and we find that quasihyperbolic geodesics in Ω are quasihyperbolic quasigeodesics in Ω_ρ; that is, a geodesic in (Ω, k) will be a quasigeodesic in (Ω_ρ, k_ρ) (the quasihyperbolization of Ω_ρ).

We remind the reader of the following basic estimates for quasihyperbolic distance, first established by Gehring and Palka [GP76, Lma.2.1]:

$$k(x,y) \ge \log\left(1 + \frac{\ell(x,y)}{d(x) \wedge d(y)}\right) \ge j(x,y) = \log\left(1 + \frac{|x-y|}{d(x) \wedge d(y)}\right) \ge \left|\log\frac{d(x)}{d(y)}\right|.$$

See also [BHK01, (2.3),(2.4)]. The first inequality above is a special case of the more general (and easily proved) inequality,

$$\ell_k(\gamma) \ge \log\left(1 + \frac{\ell(\gamma)}{d(x) \wedge d(y)}\right)$$

which holds for any rectifiable curve γ with endpoints x, y.

An immediate consequence of the above inequalities is that the identity map $(\Omega, k) \to (\Omega, d)$ is continuous; indeed,

$$B_k(x; R) \subset (e^R - 1)B(x) \qquad \text{for all } x \in \Omega \text{ and all } R > 0,$$

where $B_k(x;R)$ denotes the R-ball centered at x in (Ω,k). It is important to know when this map will be a homeomorphism (which, according to [BHK01, Lma.A.4,p.92], will be the case if and only if the identity map $(\Omega,\ell)\to(\Omega,d)$ is a homeomorphism). The following provides quantitative information concerning this question; it is easy to verify via simple estimates for the quasihyperbolic length of the 'promised short arcs'.

2.7. **Lemma.** *Suppose that (Ω,d) is a locally a-quasiconvex abstract domain. Then for all $x\in\Omega$ and all $R>0$,*

$$\tau B(x)\subset B_k(x;R)\qquad \textit{provided}\ 0\le\tau\le\min\{\lambda,R/[a(1+R)]\}.$$

The Harnack inequality (H), as stated in subsection 2.C, only requires that ρ be essentially constant on Whitney type balls. We can do the usual chaining type arguments to see that such a density ρ will satisfy a Harnack type inequality on much bigger sets, of course with a change in the Harnack constant. Here is a useful example of this phenomena.

2.8. **Lemma.** *Let ρ be a Harnack density on an abstract domain (Ω,d). If $x,y\in\Omega$ satisfy $k(x,y)\le K$, then $1/H_1\le\rho(y)/\rho(x)\le H_1$, where $H_1=H_1(K,H_\rho,\lambda)$.*

We conclude this subsection with a covering lemma for quasihyperbolic geodesics.

2.9. **Lemma.** *Suppose that (Ω,d) is a locally a-quasiconvex abstract domain. Let γ be a quasihyperbolic geodesic segment or ray in Ω with endpoint x_0. Let $x_0,x_1,x_2,\dots$ be successive points along γ with $k(x_i,x_{i-1})=K\le\log(1+\tau)$ where $\tau=\min\{\lambda,1/2a\}$. Then the balls $B_i=\tau B(x_i)$ cover γ and have bounded overlap: $\sum\chi_{B_i}\le C\chi_{\cup B_i}$, where $C=1+4/K$.*

2.E. **Uniform Spaces.** Following [BHK01], a *uniform space* is an abstract domain (so, a locally compact, rectifiably connected and non-complete metric space) (Ω,d) with the property that there is some constant $a\ge1$ such that each pair of points can be joined by an a-uniform arc. A rectifiable curve γ joining x,y in Ω is an *a-uniform arc* provided

$$\ell(\gamma)\le a|x-y|$$

and

$$\min\{\ell(\gamma(x,z)),\ell(\gamma(y,z))\}\le a\,d(z)\qquad\text{for all } z\in\gamma.$$

Here $\ell(\gamma)$ is the arclength of γ and $\gamma(x,z)$ denotes the subarc of γ between x,z. The second inequality above ensures that Ω contains the twisted double cone $\cup\{B(z;\ell(z)/a):z\in\gamma\}$ where $\ell(z)$ denotes the left-hand-side of this inequality; the first inequality asserts that this twisted double cone is not too 'crooked'. Consequently, we call γ a *double a-cone arc* if it satisfies the second inequality above (the phrases *cigar arc* and *corkscrew* are also used). Uniform domains in Euclidean space were first studied by John [Joh61] and Martio and Sarvas [MS79] and their importance in function theory is well documented; see [Geh87], [Väi88]. More recently, uniform subdomains of the Heisenberg groups, as well as more general Carnot groups, have become a focus of study; see [CT95], [CGN00], [Gre01].

An important, and characteristic, property of uniform spaces is that quasihyperbolic geodesics are uniform arcs. (See [GO79, Thms.1,2] for domains in Euclidean space and [BHK01, Thm.2.10] for general metric spaces.) With this in mind, we agree that the phrase *a-uniform space* shall designate a uniform space in which the quasihyperbolic geodesics are a-uniform arcs. Slight alterations to the proof of [BHK01, Thm.2.10] yield the following generalization of this property.

2.10. **Fact.** *In an a-uniform space, quasihyperbolic c-quasigeodesics are b-uniform arcs where $b = b(a, c)$.*

We need to know that boundary points in a uniform space can be joined by quasihyperbolic geodesics, and that these geodesics are still uniform arcs. This is a consequence of the fact that a uniform space Ω has a proper, geodesic, and Gromov hyperbolic quasihyperbolization (Ω, k) (see [BHK01, p.10, Prop.2.20, Thm.3.6]). A straightforward application of the Arzela-Ascoli theorem (cf. [Mun00, 47.1, p.290 & 46.2, p.283]) establishes the existence of quasihyperbolic geodesic rays joining an interior point to a boundary point (e.g., see [BHK01, Prop.3.12]), however, the construction for joining two boundary points with a quasihyperbolic geodesic line is a tad more delicate. It is not hard to modify and apply the proof of [BH99, Lma.3.2,p.428]; alternatively, we can simply use this result in conjunction with the fact that uniform spaces are Gromov hyperbolic. It is then easy to verify that these limit geodesics are uniform arcs.

Another crucial piece of information is a characterization of uniformity due to Gehring and Osgood [GO79, Theorems 1,2]; Bonk, Heinonen, and Koskela [BHK01, Lma.2.13] verified the necessity of this condition for the metric space setting, while the Gehring-Osgood argument can be modified to establish the sufficiency. Recalling the basic estimates for quasihyperbolic distance, we see that uniform spaces are precisely those abstract domains in which the quasihyperbolic distance is bilipschitz equivalent to the j distance. We tacitly utilize this information in what follows.

2.11. **Fact.** *A locally compact, rectifiable, non-complete metric space is a-uniform if and only if $k(x, y) \le b\, j(x, y)$ for all points x, y. The constants a and b depend only on each other. One can even take $b = 4a$ (when quasihyperbolic geodesics are a-uniform arcs).*

We conclude this subsection with some information regarding bounded uniform spaces.

2.12. **Lemma.** *Suppose Ω is a bounded a-uniform space.*

(a) *There exists $w \in \Omega$ with $d(w) = \max_{x\in\Omega} d(x)$, $1 \le \operatorname{diam}\Omega/d(w) \le 2a$, and for all points $y \in \bar{\Omega}$,*
$$|x - y| \le 4a^2 d(x) \qquad \text{for all } x \in [w, y].$$
(b) *For $0 < \lambda < 1$, all points $x \in \Omega$, and all $r \ge (\lambda/2)d(x)$, there is a point $z \in \Omega$ such that*
$$(\lambda/5)B(z) \subset B(x; r) \quad \text{and} \quad B(x; 2r) \subset (9a/\lambda)B(z).$$

Proof. Since $\bar{\Omega}$ is compact [BHK01, Prop.2.20], it is easy to obtain such a point w. If $x, y \in \Omega$, and z is the arclength midpoint of $[x, y]$, then $\ell(x, y) \le 2ad(z) \le 2ad(w)$; thus in fact $\operatorname{diam}_\ell \Omega \le 2ad(w)$. For the rest of (a): Fix $y \in \bar{\Omega}$, let $[w, y]$ be a quasihyperbolic geodesic joining y, w, and let $x \in [w, y]$. If $\ell[x, y] \le \ell[x, w]$, then $|x - y| \le \ell[x, y] \le ad(x)$. Assume that $\ell[x, y] > \ell[x, w]$. If $|x - w| \le d(w)/2$, then $d(x) \ge d(w)/2$ so $|x - y| \le \operatorname{diam}\Omega \le 2ad(w) \le 4ad(x)$. Finally, if $|x - w| > d(w)/2$, then $ad(x) \ge \ell[x, w] \ge |x - w| \ge d(w)/2$, so $|x - y| \le \operatorname{diam}\Omega \le 4a^2 d(x)$.

To see that (b) holds: Fix $0 < \lambda < 1$, $x \in \Omega$ and $r \ge (\lambda/2)d(x)$. If $|w - x| \le \lambda r/2$, then $z = w$ has the asserted properties. Suppose $|w - x| \ge \lambda r/2$. Then the point $z \in [x, w]$ with $\ell[x, z] = \lambda r/4$ does the job. $\square$

2.13. **Lemma.** *Let ρ be a Harnack Koebe density on a bounded a-uniform space (Ω, d). Suppose that Ω_ρ is bounded and that quasihyperbolic geodesics in Ω are double a-cone arcs* *in Ω_ρ. Then for any positive constant C,*

$$\operatorname{diam}_\rho[CB(z)] \simeq d_\rho(z) \qquad \textit{for all } z \in \Omega,$$

where the constant depends only on $C, H_\rho, K_\rho, a, \lambda$ and the quantity q given in the proof.

Proof. Choose $w \in \Omega$ with $d(w) = \max_{x\in\Omega} d(x)$; then $1 \le \operatorname{diam}\Omega/d(w) \le 2a$. Set $q = \operatorname{diam}_\rho(\Omega)/d_\rho(w)$; then $q \ge 1$, $d_\rho(x) \le q d_\rho(w)$, and $\ell_\rho[x,y] \le 2aq d_\rho(w)$ for all $x, y \in \Omega$. We claim that for all points $y \in \bar\Omega$,

$$\ell_\rho[x,y] \le b\, d_\rho(x) \qquad \text{for all } x \in [w,y],$$

where $b = 2a^2 qHK^2/\min\{\lambda, 1-\lambda\}$ (cf. Lemma 2.12(a)). We can assume that $\ell_\rho[x,y] > \ell_\rho[x,w]$. If $|x-w| \le \lambda d(w)$, then the Harnack and Koebe properties of ρ yield $d_\rho(x) \ge [(1-\lambda)/HK^2]d_\rho(w)$ which produces the desired assertion. On the other hand, if $|x-w| > \lambda d(w)$, then the cone arc condition gives $a d_\rho(x) \ge \ell_\rho[x,w] \ge (\lambda/HK)d_\rho(w)$.

Now fix $z \in \Omega$. If $d(z) = d(w)$, then by Fact 2.11 and Lemma 2.8, the Harnack and Koebe properties of ρ produce $\rho(w) \simeq \rho(z)$ (with a constant $H_1 = H_1(H, a, \lambda)$) and $d_\rho(w) \le H_1 K^2 d_\rho(z)$, so $\operatorname{diam}_\rho[CB(z)] \le q d_\rho(w) \le qH_1K^2 d_\rho(z)$. Assume $d(z) < d(w)$.

Let $y \in CB(z)$. If $d(y) \ge d(z)$, then Lemma 2.8 gives us $\rho(x) \simeq \rho(z)$ for all $x \in [y,z]$ (with a constant $H_2 = H_2(C, H, a\lambda)$) from which we easily conclude that $d_\rho(y,z) \le aCH_2K d_\rho(z)$. Now assume that $d(y) < d(z)$. Select $x \in [y,w]$ with $d(x) = d(z)$. Using Lemma 2.12(a) we deduce that $|x-z| \le (4a^2 + C)d(z)$, so another appeal to Lemma 2.8 gives us $\rho(u) \simeq \rho(z)$ for all $u \in [x,z]$ (with a constant $H_3 = H_3(C, H, a, \lambda)$); thus, $d_\rho(x) \le H_3K^2 d_\rho(z)$, and also $d_\rho(x,z) \le a(4a^2 + C)H_3 d_\rho(z)$. From the beginning of our proof we have that $d_\rho(y,x) \le \ell_\rho[y,x] \le b d_\rho(x)$, and therefore $d_\rho(y,z) \le d_\rho(y,x) + d_\rho(x,z) \le H_3K[bK + a(4a^2 + C)]d_\rho(z)$ as desired. □

2.F. **Uniformization.** Bonk, Heinonen and Koskela demonstrated that the quasihyperbolization (which is always proper and geodesic) of a uniform space is Gromov hyperbolic, and conversely, they established that every proper, geodesic, Gromov hyperbolic space can be conformally deformed, via *dampening* by a natural metric-density, to a bounded uniform space; see [BHK01, Prop.4.5, Prop.2.8, Thm.3.6]. (In fact they establish far more than we mention here.) We briefly describe their theory in the special case which concerns us.

We consider an abstract domain (Ω, d) with the property that the identity map $(\Omega, \ell) \to (\Omega, d)$ is a homeomorphism (so the identity $(\Omega, k) \to (\Omega, d)$ is also a homeomorphism) and such that its quasihyperbolization (Ω, k) is a Gromov hyperbolic space. According to the BHK theory, the space (Ω, k) admits a uniformizing density (called dampening) of the form

$$\rho_\varepsilon(x) = \exp(-\varepsilon k(x,w));$$

here $w \in \Omega$ is a fixed base point and $\varepsilon > 0$ a sufficiently small parameter. More precisely, if (Ω, k) is Δ-hyperbolic, then for all $0 < \varepsilon < \varepsilon(\Delta)$, the quasihyperbolic geodesics in Ω are 20-uniform arcs in (Ω, d_ε). (A careful check of BHK shows that $\varepsilon(\Delta) = [42(5 + 192\Delta + 1920\Delta^2)]^{-1}$. :-)) Here d_ε stands for the distance function obtained by conformally deforming k via the metric-density ρ_ε. Since k was obtained from the original distance function d via the quasihyperbolic density $1/d$, $\Omega_\varepsilon = (\Omega, d_\varepsilon)$ is a conformal deformation of (Ω, d) via the metric-density

$$\pi_\varepsilon(x) = \rho_\varepsilon(x)/d(x) = d(x)^{-1}\exp(-\varepsilon k(x,w));$$

again, π_ε will be a uniformizing density when $0 < \varepsilon < \varepsilon(\Delta) = (300\max\{1, \Delta\})^{-2}$.

We pause before proceeding further to record the following information relating $\partial_G\Omega$ and $\partial_\varepsilon\Omega$; see [BHK01, 3.6,3.12,4.13] and subsection 4.B.

2.14. **Fact.** *There is a natural bijection $\partial_G\Omega \to \partial_\varepsilon\Omega$ which is quasisymmetric as a map $(\partial_G\Omega, \partial) \to (\partial_\varepsilon\Omega, d_\varepsilon)$, when $0 < \varepsilon < \varepsilon(\Delta)$ and ∂ is any canonical metric on $\partial_G\Omega$. Thus the canonical conformal gauge on $\partial_G\Omega$ is naturally QS equivalent to the conformal gauge determined by d_ε on $\partial_\varepsilon\Omega$.*

We wish to know when π_ε will be a Harnack or Koebe density. We require the following information concerning ρ_ε (see [BHK01, (4.4),(4.6),(4.17)]):

$$e^{-\varepsilon k(x,y)} \le \frac{\rho_\varepsilon(x)}{\rho_\varepsilon(y)} \le e^{\varepsilon k(x,y)}, \tag{2.15}$$

$$\frac{\rho_\varepsilon(x)}{e\varepsilon} \le d_\varepsilon(x) \le (2e^{\varepsilon\kappa} - 1)\frac{\rho_\varepsilon(x)}{\varepsilon}. \tag{2.16}$$

The first set of inequalities (2.15) hold for all points $x, y \in \Omega$ and all $\varepsilon > 0$. They guarantee that the identity map $(\Omega, k) \to (\Omega, d_\varepsilon)$ is locally bilipschitz, so (Ω, d_ε) is locally compact and rectifiably connected. On the other hand, $\Omega_\varepsilon = (\Omega, d_\varepsilon)$ is non-complete, so we can form $\bar\Omega_\varepsilon$, put $\partial_\varepsilon\Omega = \bar\Omega_\varepsilon \setminus \Omega$ and let $d_\varepsilon(x) = \operatorname{dist}_\varepsilon(x, \partial_\varepsilon\Omega)$. (See [BHK01, pp.27-28]). We find that the leftmost inequality in (2.16) holds for all $x \in \Omega$ and any $\varepsilon > 0$. However, in order to obtain the rightmost inequality in (2.16), we must further require that (Ω, k) be roughly κ-starlike.

Now using (2.15), and obvious inequalities for $1/d$, we deduce that

$$e^{-2R} \le e^{-(\varepsilon+1)R} \le \frac{\pi_\varepsilon(y)}{\pi_\varepsilon(x)} \le e^{(\varepsilon+1)R} \le e^{2R}$$

for all points x, y with $k(x, y) \le R$. According to Lemma 2.7, when (Ω, d) is locally a-quasiconvex we have

$$\tau B(x) \subset B_k(x; R) \qquad \text{provided } 0 < \tau \le \min\{\lambda, R/[a(1 + R)]\}.$$

Taking $R = a$ (say) and $\tau = \min\{\lambda, 1/2a\}$ we find that for all $x \in \Omega$ and all $y \in \tau B(x)$,

$$e^{-2a} \le e^{-(\varepsilon+1)a} \le \frac{\pi_\varepsilon(y)}{\pi_\varepsilon(x)} \le e^{(\varepsilon+1)a} \le e^{2a};$$

that is, π_ε is a Harnack density with constants $H = e^{2a}$ and τ which are independent of ε.

Finally, since Ω_ε is non-complete, we can ask whether or not π_ε is a Koebe density. Since $\pi_\varepsilon(x) = \rho_\varepsilon(x)/d(x)$, we see from (2.16) that π_ε will be a Koebe density, with constant $K = (2e^{\varepsilon\kappa} - 1)/\varepsilon$ (assuming $\varepsilon e \le 1$), provided (Ω, k) is roughly κ-starlike (and (Ω, d) uniformly locally quasiconvex).

These conditions describing when π_ε will be a Harnack or Koebe density do not require that (Ω, k) be Gromov hyperbolic. We record the above information for later reference; see also Lemma 2.2 and Corollary 3.3. Note too that (Ω, d_ε) is bounded with $\operatorname{diam}_\varepsilon \Omega_\varepsilon \le 2/\varepsilon$.

2.17. **Lemma.** *Let (Ω, d) be a locally a-quasiconvex abstract domain and fix a base point $w \in \Omega$. Then for any $\varepsilon > 0$,*

$$\pi_\varepsilon(x) = \rho_\varepsilon(x)/d(x) = d(x)^{-1}\exp(-\varepsilon k(x, w))$$

is a Harnack density with constant $H = e^{2a}$. If in addition (Ω, k) is roughly κ-starlike, then π_ε is also a Koebe density, with constant $K = \max\{\varepsilon e, (2e^{\varepsilon\kappa} - 1)/\varepsilon\}$.

We point out that the above, in conjunction with Lemmas 2.6 and 3.1(b), provides a one-to-one correspondence between conformal metric-densities on Ω and the same on Ω_ε. We record a precise statement of this for later use.

2.18. **Corollary.** *Suppose (Ω, d, μ) is an abstract domain having bounded geometry and a Gromov hyperbolic roughly starlike quasihyperbolization (Ω, k). Let $\Omega_\varepsilon = (\Omega, d_\varepsilon, \mu_\varepsilon)$ be the deformation of Ω via the density π_ε defined just above. If σ is a conformal density on Ω_ε, then its pull-back $\rho = \sigma\,\pi_\varepsilon$ is a conformal density on Ω, and conversely if ρ is a conformal density on Ω, then its push-forward $\sigma = \rho\,\pi_\varepsilon^{-1}$ is a conformal density on Ω_ε. In both cases $\Omega_\rho = (\Omega_\varepsilon)_\sigma$, and the metric-density parameters depend only on each other and the data associated with Ω.*

3. Proofs of Main Results

Here we define and discuss bounded geometry and doubling conformal densities. Then we corroborate the results asserted in the Introduction. We also present a local doubling property of quasihyperbolic space which may be of independent interest. Note that when Ω is the Euclidean unit ball, Theorems B,C,D are [BHR01, Props.2.11,1.25,1.28].

In each of our main theorems (B,C,D, and the sufficiency part of A), all of which involve the existence of certain conformal densities, we assume the basic minimal hypotheses that (Ω, d, μ) is an abstract domain having bounded geometry and a Gromov hyperbolic roughly starlike quasihyperbolization (Ω, k). According to Corollary 2.18, there is then a one-to-one correspondence between conformal metric-densities on Ω and the same densities on any of the uniform spaces Ω_ε obtained from Ω via the densities π_ε discussed in subsection 2.F. The upshot of this discussion is that we can utilize the Bonk-Heinonen-Koskela uniformization theory to pass to the space Ω_ε, which is a bounded uniform locally Ahlfors regular locally Loewner space when $\varepsilon < \varepsilon(\Delta)$ (see Corollary 3.3). Thus we may as well assume from the onset that Ω is a bounded uniform locally Ahlfors Q-regular locally Q-Loewner space (say, with parameters M and φ which depend only on the original data M, m, λ, Δ, κ associated with Ω).

The observant reader might raise a potentially serious objection to our proposed course of action: The quasihyperbolic geodesics in Ω may not be quasihyperbolic geodesics in Ω_ε. However, since the uniformizing density π_ε is Koebe, the identity map $(\Omega, k) \to (\Omega_\varepsilon, k_\varepsilon)$, on the quasihyperbolizations of Ω and Ω_ε, will be bilipschitz. Therefore, quasihyperbolic geodesics in Ω will be quasihyperbolic quasigeodesics in Ω_ε, so in particular, they will be a-uniform arcs (see Fact 2.10) where $a = a(\Delta, \kappa)$.

3.A. Bounded Geometry and its Consequences.

Recall our definition that an abstract domain (Ω, d, μ) is a locally compact, rectifiably connected, non-complete metric measure space; these are our standing metric hypotheses. We say that (Ω, d, μ) has *bounded Q-geometry*, $Q > 1$, provided it is both *locally upper Ahlfors Q-regular* (see subsection 4.D) and *weakly locally Q-Loewner*; this latter condition means that there exists a positive constant m such that for all $x \in \Omega$, and all non-degenerate disjoint continua E, F in $\lambda B(x)$,

$$\Sigma(E,F) \le 16 \implies \quad \operatorname{mod}_Q(E,F;\Omega) \ge m.$$

Any space Ω with the property that Whitney balls $\lambda B(x)$ (with a fixed parameter) are uniformly bilipschitz equivalent to Euclidean balls (in a fixed dimension) is easily seen to have bounded geometry. Other examples include Riemannian manifolds with Ricci bounded geometry as well as the exotic examples of Bourdon-Pajot and Laakso for any $Q > 1$; see [BHK01, Exs.9.7,p.86] and the references mentioned there. (Note that by Lemma 3.1, (Ω, d, μ) has bounded geometry if and only if its quasihyperbolization satisfies the condition studied in [BHK01, Chpt.9].)

Using Lemma 2.2 it is straightforward to show that if Ω is a locally a-quasiconvex abstract domain, and there exists some conformal density ρ on Ω, then in fact Ω is locally upper Ahlfors Q-regular with constant $M = (aH_\rho^2 K_\rho^2)^Q A_\rho$. Thus the first part of bounded Q-geometry is a necessary condition for Ω to support any conformal density (at least when Ω is locally quasiconvex). The second part of bounded Q-geometry, the weak local Loewner criterion, can also be described in terms of Poincaré inequalities as explained in [BHK01, Prop.9.4] and [HK98, §5]; it ensures that there are plenty of curves available (e.g., it gives local quasiconvexity). However, to substantiate this existence of many curves requires the use of certain modulus estimates (see Facts 4.3,4.4), and these estimates in turn require an upper mass condition.

The Loewner 'part' of bounded Q-geometry also plays an essential role in two other places. First, it is a key player in the proof of Proposition 2.4, which is the crucial ingredient in the proof of (d) implies (e) in Theorem B. Second, for uniform Loewner spaces with appropriately 'thick' boundaries, the Koebe condition for a metric-density follows from the Harnack and Ahlfors condition; this fact is utilized in the proof of Theorem C.

Bounded geometry provides a number of essential properties for our underlying space. Our arguments are modifications of standard ones.

3.1. **Lemma.** *Let (Ω, d, μ) be an abstract domain having bounded Q-geometry (with constants M, m, λ). Then:*

(a) *μ is locally Ahlfors Q-regular.*
(b) *Ω is locally quasiconvex with constants which depend only on Q, M, m, λ.*
(c) *Ω is locally Q-Loewner with a control function ψ and parameters κ, ε_0 which depend only on the data Q, M, m, λ associated with Ω.*

Proof. **(a)** Fix $x \in \Omega$ and $0 < r < \lambda d(x)$. We verify that $\mu[B(x;r)] \geq m(r/3)^Q$. Choose a point $y \in \Omega \setminus B(x;r)$ and an arc α joining x, y. Let E be the x-component of $\alpha \cap \bar{B}(x;r/3)$ and let F be a subcontinuum of α which joins the spheres $S(x;2r/3), S(x;r)$ inside the ring $B(x;r) \setminus \bar{B}(x;2r/3)$ (cf. [Kur68, Thm.1, p.172] or [HY88, Thm.2-16, p.47]). Then $\Sigma(E,F) \leq 2$, so $\operatorname{mod}_Q(E,F) \geq m$.

Since each arc joining E, F has length at least $r/3$, the density defined by $\rho(z) = 3/r$ for $z \in B(x;r)$, and $\rho = 0$ elsewhere, is admissible for $\Delta(E,F)$. We conclude that

$$m \leq \int \rho^Q \, d\mu = (3/r)^Q \mu[B(x;r)].$$

(b) Since a (local) upper mass condition holds for μ, we can utilize the spherical ring estimate (Fact 4.4) to assert that

$$\operatorname{mod}_Q \Gamma \leq N(\log 16\kappa)^{1-Q}$$

for any family Γ of curves which join the spheres $S(x;r/16), S(x;\kappa r)$, provided $\kappa \geq 1/8$ and $2\kappa r \leq \lambda d(x)$; here $N = 2^{Q+1}M/\log 2$. Assume $m \leq 1 \leq M$ and define

$$\kappa = \exp[(2N/m)^{1/(Q-1)}] \geq 2 \quad \text{and} \quad \sigma = \lambda/5\kappa.$$

Notice that $N(\log 16\kappa)^{1-Q} \leq m/2$.

Fix $z \in \Omega$. Let $x, y \in \sigma B(z)$ and write $r = |x - y|$. We construct a rectifiable arc α joining x, y with $\ell(\alpha) \leq a|x - y|$. Since $d(x) \geq (1 - \sigma)d(z) \geq (4/5)d(z)$, $2\kappa r \leq 4\kappa\sigma d(z) \leq \lambda d(x)$ and thus we can employ the modulus estimate described above. We use an idea due to Gehring-Martio; see their proof that quasiextremal distance domains are quasiconvex [GM85, Thm.2.7].

Let E, F be continua (e.g. arcs) joining the points x, y to the spheres $S(x; r/16)$, $S(y; r/16)$ respectively. As $\Sigma(E, F) \leq 16$, $\operatorname{mod}_Q(E, F) \geq m$. Let $\Gamma = \Delta(E, F) \setminus \Delta(E, F; B(x; \kappa r))$; then each curve in Γ joins the spheres $S(x; r/16)$, $S(x; \lambda r)$. We now have

$$m \leq \operatorname{mod}_Q(E, F) \leq \operatorname{mod}_Q(E, F; B(x; \kappa r)) + \operatorname{mod}_Q \Gamma \leq \operatorname{mod}_Q(E, F; B(x; \kappa r)) + m/2,$$

and thus $\operatorname{mod}_Q(E, F; B(x; \kappa r)) \geq m/2$.

Suppose each curve $\gamma \in \Delta(E, F; B(x; \lambda r))$ has arclength $\ell(\gamma) \geq L = Cr$ for some $C > 0$. Appealing to a standard modulus estimate (Fact 4.3) we can assert that

$$m/2 \leq \operatorname{mod}_Q(E, F; B(x; \kappa r)) \leq M(\kappa/C)^Q,$$

which means that $C \leq \kappa(2M/m)^{1/Q}$. Therefore there is a curve $\gamma_0 \in \Delta(E, F; B(x; \kappa r))$ with $\ell(\gamma_0) \leq c\, r$ where $c = \kappa(2M/m)^{1/Q}$.

We iterate the above procedure. Choose a point $y_1 \in \gamma_0 \cap E$, put $r_1 = |x - y_1|$, and let E_1, F_1 be the $x-$, y_1-components of $E \cap \bar{B}(x; r_1/16)$, $\gamma_0 \cap \bar{B}(y_1; r_1/16)$ respectively. Arguing as above we deduce the existence of a curve $\alpha_1 \in \Delta(E_1, F_1; B(x; \kappa r_1))$ with $\ell(\alpha_1) \leq c\, r_1 \leq c\, r/16$. Continuing this process, we obtain curves α_n which join α_{n-1} to $E \cap \bar{B}(x; r/16^n)$ and have arclength $\ell(\alpha_n) \leq c\, r/16^n$. In a similar manner we obtain arcs β_1 joining γ_0 to F and β_n joining β_{n-1} to $F \cap \bar{B}(y; r/16^n)$ with $\ell(\beta_n) \leq c\, r/16^n$. Then the connected set $\bigcup \alpha_n \cup \gamma_0 \cup \bigcup \beta_n$ joins x, y and has $\mathcal{H}^1$-measure at most $cr + 2cr \sum 16^{-n} \leq 2cr$. An appeal to, e.g., [Fal87, 3.12] provides the desired curve with arclength at most $a|x - y|$, $a = 2c$.

(c) To verify that Ω is locally Q-Loewner we use the criterion given in [BHK01, Prop.6.49]. Note that their proof only requires uniform local quasiconvexity (see [BHK01, (6.53), p.59]), which we have thanks to (b) above. Thus we must produce constants $n > 0$, $\kappa \geq 1$, and $0 < \varepsilon_0 \leq 1/\kappa$ with the property that: for all $0 < \varepsilon \leq \varepsilon_0$, all $z \in \Omega$, and all non-degenerate disjoint continua E, F in $\varepsilon B(z)$,

$$\Sigma(E, F) \leq 16 \implies \quad \operatorname{mod}_Q(E, F; \kappa\varepsilon B(z)) \geq n.$$

Again, as in the proof of (b) above, assume $m \leq 1 \leq M$ and define $\kappa = \exp[(2N/m)^{1/(Q-1)}]$ and $\varepsilon_0 = \sigma = \lambda/5\kappa$, where $N = 2^{Q+1}M/\log 2$; these parameters depend only on Q, M, m, λ.

Fix $z \in \Omega$. Then by (b) and our choice of ε_0 we know that points in $\varepsilon_0 B(z)$ can be joined by 'short' arcs. Also as in the proof of (b), we find that

$$\operatorname{mod}_Q \Gamma \leq N(\log \kappa)^{1-Q} = m/2$$

for any family of curves which join the spheres $S(z; \varepsilon d(z))$, $S(z; \kappa\varepsilon d(z))$ provided $\kappa\varepsilon \leq \lambda$ (which is true for all $0 < \varepsilon \leq \varepsilon_0$).

Fix $0 < \varepsilon \leq \varepsilon_0$ and suppose non-degenerate disjoint continua E, F in $\varepsilon B(z)$ satisfy $\Sigma(E, F) \leq 16$. Put $\Gamma = \Delta(E, F) \setminus \Delta(E, F; \kappa\varepsilon B(z))$, so each curve in Γ joins the spheres $S(z; \varepsilon d(z))$, $S(z; \kappa\varepsilon d(z))$. Then, since Ω is a weak locally Q-Loewner space, we find via the above that

$$m \leq \operatorname{mod}_Q(E, F) \leq \operatorname{mod}_Q(E, F; \kappa\varepsilon B(z)) + \operatorname{mod}_Q \Gamma \leq \operatorname{mod}_Q(E, F; \kappa\varepsilon B(z)) + m/2$$

and therefore $\operatorname{mod}_Q(E, F; \kappa\varepsilon B(z)) \geq m/2$ as desired. □

Here is a useful consequence of Lemma 3.1.

3.2. **Corollary.** *Suppose ρ is a Harnack Koebe density on an abstract domain (Ω, d, μ) having bounded Q-geometry. Then Ω_ρ is locally Ahlfors Q-regular and locally Q-Loewner with parameters and a control function which depend only on the data associated with ρ and Ω.*

Proof. Thanks to Lemma 3.1, we know that Ω is locally Ahlfors Q-regular, locally a-quasiconvex and locally Q-Loewner. Corollary 2.3 now tells us that for all $x \in \Omega$ and all $0 \le r \le (\lambda/2a^2H)\rho(x)d(x)$,

$$m(3aH^2)^{-Q}r^Q \le \mu_\rho[B_\rho(x;r)] \le H^{2Q}Mr^Q;$$

here $H = H_\rho$, M, m are the bounded geometry constants, and λ is the minimum of all the various Whitney ball constants. Since ρ is also Koebe, we easily deduce from the above that Ω_ρ is locally Ahlfors Q-regular with constants which depend only on Q, M, m, H, K, λ where $K = K_\rho$.

To verify that Ω_ρ is locally Q-Loewner we appeal to [BHK01, Prop.6.39]; their proof only requires uniform local quasiconvexity (see [BHK01, (6.42), p.56]). □

Note that if Ω_ρ above is also uniform, then it would be globally Loewner by [BHK01, Thm.6.4]. We record the following consequence of this observation.

3.3. **Corollary.** *Let (Ω, d, μ) be an abstract domain having bounded Q-geometry (with constants M, m) and a Gromov Δ-hyperbolic roughly κ-starlike quasihyperbolization (Ω, k). Then any BHK-uniformization $\Omega_\varepsilon = (\Omega, d_\varepsilon, \mu_\varepsilon)$ of Ω is a bounded locally Ahlfors Q-regular locally Q-Loewner space (hence of bounded Q-geometry), and even Q-Loewner when $\varepsilon < \varepsilon(\Delta)$. Here the various parameters and control functions depend only on $\Delta, \kappa, Q, M, m, \lambda, \varepsilon$.*

Proof. That Ω_ε is locally Ahlfors Q-regular and locally Q-Loewner follows from Corollary 3.2 because the metric-density π_ε is Harnack and Koebe (see Lemma 2.17). If $\varepsilon < \varepsilon(\Delta)$, then Ω_ε is uniform and so the local Loewner condition implies a global one. □

3.B. **Doubling Conformal Densities.** Following [BHR01], we say that a metric-density ρ, on a *uniform* space (Ω, d, μ), is *doubling* provided μ_ρ is a doubling measure on (Ω, d); i.e., there exists a constant $D = D_\rho$ such that for all $x \in \Omega$,

$$\text{(D)} \qquad \mu_\rho[B(x;2r)] \le D\,\mu_\rho[B(x;r)] \qquad \text{for all } r > 0.$$

When Ω is not uniform, the above doubling condition may fail to hold even if ρ is 'nice'; e.g., consider $\rho = |f'|$ on $\Omega = \{x + iy : |y| < 1\}$ in the complex plane, where f is a conformal homeomorphism of Ω onto the unit disk. To compensate for this we employ the following definition: a conformal density ρ, on an abstract domain (Ω, d, μ) with Gromov hyperbolic quasihyperbolization, is *doubling* if its push-forward is doubling on some BHK-uniformization Ω_ε of Ω. (Since all such spaces Ω_ε are QS equivalent, and doubling measures can be defined for conformal gauges, there is no ambiguity here. See §4.A for a discussion of conformal gauges, and also the very last paragraph of §4.E.)

Notice that this doubling condition (D), which uses d-balls but μ_ρ-measure, interweaves the measure properties of the deformed space with the metric properties of the original (or uniformized) space.

The following two results, along with Proposition 2.4, provide the essential ingredients for our proof of Theorem B; these ideas (modeled after [BHR01, Prop.2.5]) may be of independent interest. First we present a necessary condition for a metric-density to be doubling.

3.4. **Proposition.** *Let ρ be a Harnack density on an a-uniform locally Ahlfors Q-regular space (Ω, d, μ) (with constants H, M, λ). Suppose that ρ is also doubling on Ω (with constant D). Then quasihyperbolic geodesics in Ω are double c-cone arcs in Ω_ρ where $c = c(D, H, M, Q, a, \lambda)$. If in addition Ω is bounded, then so is Ω_ρ and $\partial\Omega \subset \partial_\rho\Omega$, with equality holding when Ω is also locally Q-Loewner.*

Proof. Fix $x, y \in \bar{\Omega}$, $z \in [x, y]$, and assume that $\ell[x, z] \le \ell[y, z]$. We show that $\ell_\rho[x, z] \le c\, d_\rho(z)$. Put $\tau = \lambda/2a$. Choose successive points $z_0 = z, z_1, \ldots$ along $[z, x]$, with $k(z_i, z_{i-1}) = \log(1 + \tau)$ for all i and $z_i \to x$ when $x \in \partial\Omega$, or with $k(z_i, z_{i-1}) = \log(1 + \tau)$ for $0 \le i < n$ and $k(x, z_{n-1}) \le \log(1 + \tau)$ when $z_n = x \in \Omega$. Let $B_i = \tau B(z_i)$. Then $H^{-1} \le \rho/\rho(z_i) \le H$ in B_i, and according to Lemmas 2.7 and 2.9

$$(z_{i-1}, z_{i+1}) \subset B_k(z_i; \log(1 + \tau)) \subset B_i \subset B_k(z_i; 1),$$

and the B_i have bounded overlap with a constant which depends only on τ. Using this information in conjunction with Ω being locally Ahlfors Q-regular we deduce that

$$(MH^Q)^{-1} \le \frac{\mu_\rho(B_i)}{[\tau\rho(z_i)d(z_i)]^Q} \le MH^Q.$$

It is not difficult to see that $\cup_{i\ge j} B_i \subset (a + 1)B_j$ for all j. (For any $u \in [x, z_j]$ we have $|u - z_j| \le \ell[x, z_j] \le ad(z_j)$, so $d(u) \le (a + 1)d(z_j)$. Thus if $u \in B_i$, then $|u - z_j| \le |u - z_i| + |z_i - z_j| < \tau d(z_i) + ad(z_j) \le [\tau(a + 1) + a]d(z_j) \le (a + 1)d(z_j)$ because $\tau(a + 1) \le \lambda(a + 1)/2a \le \lambda \le 1$.) Since the B_i have bounded overlap and μ_ρ is doubling on Ω, the above estimates on $\mu_\rho(B_i)$ yield

$$\sum_{i\ge j} [\rho(z_i)d(z_i)]^Q \le C[\rho(z_j)d(z_j)]^Q,$$

where $C = C(D, H, M, a, \lambda)$, and this holds for all $j \ge 0$. A numerical lemma (cf. [HS01, 5.1]) now gives us

$$\sum_{i\ge 0} \rho(z_i)d(z_i) \le C\rho(z)d(z),$$

where the new constant C also depends on Q. By construction of the balls B_i,

$$\ell_\rho[z_i, z_{i+1}] \le H\rho(z_i)\ell[z_i, z_{i+1}] \le aH\rho(z_i)|z_i - z_{i+1}| \le a\tau H\rho(z_i)d(z_i),$$

so we obtain

$$\ell_\rho[x, z] = \sum_{i\ge 0} \ell_\rho[z_i, z_{i+1}] \le a\tau HC\rho(z)d(z) \le a(\tau/\lambda)H^2Cd_\rho(z),$$

and thus $[x, y]$ is a double c-cone arc with $c = 2H^2C$.

In fact, a careful look at the above reveals that $\ell_\rho[x, y] \lesssim \rho(w)d(w)$ where $w \in [x, y]$ satisfies $d(w) = \max\{d(u) : u \in [x, y]\}$. Indeed, let z be the arclength midpoint of $[x, y]$. The above argument gives us $\ell_\rho[x, y] \le C\rho(z)d(z)$. Assuming that $w \in [x, z]$ we find that $|w-z| \le ad(z)$, so by Fact 2.11 $k(w, z) \le 4a \log(1 + a)$ and thus by Lemma 2.8 we have $\rho(w) \simeq \rho(z)$. (Also, if $x, y \in (\lambda/2a)B(w)$, Lemma 2.2 gives us $\ell_\rho[x, y] \ge d_\rho(x, y) \simeq \rho(w)|x - y|$; if one of x or y lies outside this Whitney ball, then $\ell_\rho[x, y] \gtrsim \rho(w)d(w)$.)

Now suppose Ω is bounded. Choose $w \in \Omega$ with $d(w) = \max_{x\in\Omega} d(x)$. From what was just said, we see that for any $x \in \Omega$, $d_\rho(x, w) \le \ell_\rho[x, w] \lesssim \rho(w)d(w)$. Thus $\mathrm{diam}_\rho(\Omega) \simeq \rho(w)d(w)$; in particular, Ω_ρ is bounded. Also, if $\xi \in \partial\Omega$, then $\ell_\rho[w, \xi) < \infty$, so $\xi \in \partial_\rho\Omega$. Finally, if Ω is locally Q-Loewner, then [Her04, Cor.6.1] asserts that $\partial_\rho\Omega = \partial\Omega$. □

Next we establish a sufficient condition for a metric-density to be doubling.

3.5. **Proposition.** *Let ρ be a conformal density on an a-uniform locally lower Ahlfors Q-regular space (Ω, d, μ) (with constants H, A, K, M, λ). Suppose that quasihyperbolic geodesics in Ω are double c-cone arcs in Ω_ρ. Then Ω_ρ is Ahlfors Q-regular with a parameter which depends only on c and the data $H, A, K, M, Q, a, \lambda$ associated with ρ and Ω. If in addition Ω and Ω_ρ are both bounded, then ρ is doubling on Ω with a parameter which depends only on the aforementioned data and the quantity q given in the proof of Lemma 2.13.*

Proof. Since Ω_ρ is assumed to be upper Ahlfors Q-regular, we need only check that μ_ρ is lower Ahlfors Q-regular. Fix a point $x \in \Omega$. The hypothesis that Ω be locally lower Ahlfors Q-regular means that

$$\mu[B(x; r)] \geq M^{-1} r^Q \qquad \text{for } 0 < r \leq \lambda d(x).$$

Then according to Corollary 2.3,

$$\mu_\rho[B_\rho(x; r)] \geq M^{-1}(r/aH^2)^Q \qquad \text{for } 0 < r \leq (\lambda/2a^2H)\rho(x)d(x).$$

We may assume that $0 < r \leq (1/2)\operatorname{diam}_\rho(\Omega)$, so there is a point $w \in \Omega$ with $d_\rho(x, w) \geq r$. Choose $z \in [x, w]$ with $\ell_\rho[x, z] = r/2$. Then $r/2c \leq d_\rho(z) \leq K\rho(z)d(z)$. Put $C = 4a^2cHK/\lambda$. Then $r/C \leq (\lambda/2a^2H)\rho(z)d(z)$ and $B_\rho(z; r/C) \subset B_\rho(x; r)$, so

$$\mu_\rho[B_\rho(x; r)] \geq M^{-1}(r/aCH^2)^Q.$$

Thus Ω_ρ is Ahlfors Q-regular with constant $N = \max\{A, M(8ca^3H^3K/\lambda)^Q\}$.

Now suppose also that Ω and Ω_ρ are both bounded. Put $C = aHK$. Lemma 2.2 asserts that for all $z \in \Omega$ and all $0 \leq \vartheta \leq \lambda/(2C)$, $C^{-1}\vartheta B(z) \subset \vartheta B_\rho(z) \subset C\vartheta B(z)$.

We proceed to verify that ρ is doubling on (Ω, d, μ). Let $x \in \Omega$, $d = d(x)$ and $r > 0$. If $r \leq [\lambda/(2C)^2]d$, then $\vartheta = 2Cr/d \leq \lambda/(2C)$, so $B(x; 2r) = (\vartheta/C)B(x) \subset \vartheta B_\rho(x)$ and therefore

$$\mu_\rho[B(x; 2r)] \leq N(\vartheta d_\rho(x))^Q = N(2C^2)^Q(\frac{\vartheta}{2C^2}d_\rho(x))^Q \leq$$
$$\leq N^2(2C^2)^Q\mu_\rho[\frac{\vartheta}{2C^2}B_\rho(x)] \leq N^2(2C^2)^Q\mu_\rho[B(x; r)]$$

where the last inequality holds because $(\vartheta/2C^2)B_\rho(x) \subset (\vartheta/2C)B(x) = B(x; r)$.

Suppose that $r \geq [\lambda/(2C)^2]d$. Appealing to Lemma 2.12(b) we find a point $z \in \Omega$ with

$$\lambda(10C^2)^{-1}B(z) \subset B(x; r) \qquad \text{and} \qquad B(x; 2r) \subset \lambda^{-1}18aC^2B(z).$$

Utilizing Lemma 2.13 we find that $B(x; 2r) \subset bB_\rho(z)$ where b depends only on q, c, a, λ, H, K. Setting $\vartheta = \lambda/(10C^3) \leq \lambda/(2C)$ we now obtain

$$\mu_\rho[B(x; 2r)] \leq N(b/\vartheta)^Q(\vartheta d_\rho(z))^Q \leq N^2(b/\vartheta)^Q\mu_\rho[\vartheta B_\rho(x)] \leq N^2(b/\vartheta)^Q\mu_\rho[B(x; r)]$$

where the last inequality holds because $\vartheta B_\rho(z) \subset C\vartheta B(z) = (\lambda/10C^2)B(z)$.

We conclude that ρ is doubling with constant $D = N^2(10bC^3/\lambda)^Q$. □

3.C. **Proof of Theorem B.** As discussed at the beginning of this Section, we may assume that (Ω, d, μ) is a bounded a-uniform locally Ahlfors Q-regular locally (and globally) Q-Loewner space. Suppose that ρ is a conformal density on Ω (with constants H, A, K). Then the Gehring-Hayman Inequality holds [Her04, Thm.A]:

$$\ell_\rho[x, y] \le \Lambda d_\rho(x, y) \qquad \text{for all } x, y \in \Omega,$$

where Λ depends only on the data associated with Ω and ρ.

We follow the line of reasoning for the proof of [BHR01, Prop.2.11].

(a) $\Longrightarrow$ **(b)** Assume ρ is doubling on Ω. By Proposition 3.4, Ω_ρ is bounded and quasihyperbolic geodesics in Ω are double c-cone arcs in Ω_ρ with $c = c(D, H, M, Q, a, \lambda)$. Thus these geodesics are b-uniform arcs in Ω_ρ, $b = \max\{c, \Lambda\}$.

(b) $\Longrightarrow$ **(c)** Suppose Ω_ρ is bounded and uniform. Since Ω_ρ is locally Q-Loewner (by Corollary 3.2) and uniform, it is Q-Loewner by [BHK01, Thm.6.4].

(c) $\Longrightarrow$ **(d)** This follow from the elementary observation that every Q-Loewner space is automatically lower Ahlfors Q-regular, and we are assuming upper Ahlfors Q-regularity. (Alternatively, we can use the first part of Proposition 3.5, which requires that quasihyperbolic geodesics in Ω be double cone arcs in Ω_ρ. Since ρ is Koebe, the identity map $(\Omega, k) \to (\Omega, k_\rho)$ is bilipschitz, so quasihyperbolic geodesics in Ω are quasihyperbolic quasigeodesics in Ω_ρ, and therefore by Fact 2.10 these are uniform arcs.)

(d) $\Longrightarrow$ **(e)** Here we assume that Ω_ρ is bounded and Q-Loewner. An appeal to Corollary 2.5 yields the desired assertion. Note that in addition to the data for Ω, ρ, and Ω_ρ, the quasisymmetry distortion function will also depend on the quantity q introduced in the proof of the Proposition 2.4; however, if we take as our base point $w \in \Omega$ with $d(w)$ maximal, then the ratio r satisfies $1 \le r \le 2a$.

(e) $\Longrightarrow$ **(a)** We assume the identity $\Omega_\rho \to \Omega$ is η-quasisymmetric. We employ the second part of Proposition 3.5 to demonstrate that ρ is doubling. This requires us to check that Ω_ρ is bounded and that quasihyperbolic geodesics in Ω are uniform arcs in Ω_ρ. Since quasisymmetric homeomorphisms map bounded spaces to bounded spaces [TV80, Thm.2.5], Ω_ρ is bounded. Also, by [TV80, Thm.2.25], the identity map extends to a quasisymmetric homeomorphism $\bar{\Omega}_\rho \to \bar{\Omega}$ and so in particular we have a quasisymmetric equivalence of $\partial_\rho\Omega$ and $\partial\Omega$.

Since the Gehring-Hayman Inequality holds, it remains to verify that quasihyperbolic geodesics in Ω are double cone arcs in Ω_ρ. Fix a quasihyperbolic geodesic $[x, y]$ in Ω, let $z \in [x, y]$, and suppose that $\ell[x, z] \le \ell[y, z]$. For each point $\zeta \in \partial\Omega$,

$$|x - z| \le \ell[x, z] \le a d(z) \le a|\zeta - z|,$$

so by quasisymmetry we obtain

$$d_\rho(x, z) \le \eta(a) d_\rho(\zeta, z).$$

Since this holds for all boundary points u, we conclude that

$$\ell_\rho[x, z] \le \Lambda d_\rho(x, z) \le \eta(a)\Lambda d_\rho(z)$$

and thus $[x, y]$ is a b-uniform arc with $b = \eta(a)\Lambda$. (Note that the quantity q appearing in Proposition 3.5 satisfies $q \le 2\eta(2a)$ and so does not enter the data for the doubling constant.) $\square$

3.D. **Quasihyperbolic Doubling.** It is well-known that Euclidean space is doubling while hyperbolic space is not; however, doubling does hold in hyperbolic space for balls with bounded radii. The following result is an analog of this for the quasihyperbolization of a locally quasiconvex locally Ahlfors regular space. It is needed below in our proof of Theorem C, and may also be of independent interest.

3.6. **Proposition.** *Let (Ω, d, μ) be a locally a-quasiconvex locally Ahlfors Q-regular abstract domain. Then its quasihyperbolization (Ω, k) is locally doubling in the sense that if $\Sigma \subset \Omega$ is a set of points satisfying*

$$0 < t \le k(x,y) \le T < \infty \qquad \text{for all } x, y \in \Sigma,\ x \ne y,$$

then the cardinality of Σ is bounded by

$$\#\Sigma \le 2M^2 8^Q (2M^2 24^Q)^{8aT/\lambda t}.$$

Here M is the local regularity constant and λ the Whitney ball constant.

Proof. Let Σ be such a set. Assume $t \le 1$ and put $\tau = \lambda t/4a$. By Lemma 2.7, $\tau B(x) \subset B_k(x; t/2)$, so $\tau B(x) \cap \tau B(y) = \emptyset$ for all points $x \ne y$ in Σ. Suppose, at first, that $z \in \Sigma \subset 2\tau B(z)$. Then for each $x \in \Sigma$,

$$(1/2)d(z) \le (1 - 2\tau)d(z) \le d(x) \le (1 + 2\tau)d(z),$$

so, $\tau B(x) \subset 4\tau B(z) \subset \lambda B(z)$, and thus by local Q-regularity

$$M\,(4\tau d(z))^Q \ge \mu[4\tau B(z)] \ge \sum_{x\in\Sigma} \mu[\tau B(x)] \ge (\#\Sigma) M^{-1}\,(\tau d(z)/2)^Q\,.$$

We conclude that

$$z \in \Sigma \subset 2\tau B(z) \implies \#\Sigma \le 8^Q M^2. \tag{3.7}$$

Now fix a point $z_0 \in \Sigma$; so $\Sigma \subset B_k(z_0; T)$. Let $y_1, y_2, \dots, y_N$ be distinct points in $\Sigma \setminus \{z_0\}$. We verify that N cannot be too large. If L of these points y_i lie in $2\tau B(z_0)$, then by the proof of (3.7) we obtain $L \le 8^Q M^2$; it follows that we may assume that 'most' (more than, say, $N/2$) of the points y_i lie outside $2\tau B(z_0)$.

Put $d_0 = d(z_0)$ and fix quasihyperbolic geodesics $[z_0, y_i]$. For each $y_i \notin 2\tau B(z_0)$, let x_{1i} be the first point along $[z_0, y_i]$ with $|x_{1i} - z_0| = 2\tau d_0$. For these points x_{1i},

$$(1/2)d_0 \le (1 - 2\tau)d_0 \le d(x_{1i}) \le (1 + 2\tau)d_0 \le (3/2)d_0,$$

and

$$k(z_0, x_{1i}) \ge \frac{\ell[z_0, x_{1i}]}{(1 + 2\tau)d_0} \ge \frac{2\tau d_0}{(1 + 2\tau)d_0} \ge \tau.$$

Thus there is a set $I_1 = \{i : y_i \notin 2\tau B(z_0)\} \subset \{1, 2, \dots, N\}$ with $\#I_1 \ge N/2$ and

$$k(z_0, x_{1i}) \ge \tau \qquad \text{for all } i \in I_1.$$

We claim that one of the points x_{1i} enjoys the property that $\tau B(x_{1i})$ contains at least $N/(2M^2 24^Q)$ of the points x_{1j} ($j \in I_1$). Indeed, suppose that for some L,

$$\sum_{I_1} \chi_{(\tau/4)B(x_{1i})} \le L.$$

Now $\cup_{I_1}(\tau/4)B(x_{1i}) \subset 3\tau B(z_0) \subset \lambda B(z_0)$, so by local Q-regularity

$$M\,(3\tau d_0)^Q \geq \mu\left(\bigcup_{i\in I_1}(\tau/4)B(x_{1i})\right) \geq L^{-1}\sum_{i\in I_1}\mu[(\tau/4)B(x_{1i})] \geq$$
$$\geq (LM)^{-1}\sum_{i\in I_1}(\tau d(x_{1i})/4)^Q \geq \frac{\#I_1}{LM}\left(\frac{\tau d_0}{8}\right)^Q.$$

Therefore, $L \geq \#I_1/(M^2 24^Q) \geq N/(2M^2 24^Q)$. Finally, since

$$(\tau/4)B(x) \cap (\tau/4)B(y) \neq \emptyset \implies y \in \tau B(x),$$

the above lower bound on the minimal possible overlap of the balls $(\tau/4)B(x_{1i})$ corroborates our assertion: there exists z_1 (one of the points x_{1i}, $i \in I_1$) and $J_1 \subset I_1$ such that

$$x_{1j} \in \tau B(z_1) \quad \text{for all } j \in J_1 \text{ and } \quad \#J_1 \geq N_1 = N/(2M^2 24^Q).$$

As above, we may assume that 'most' (at least $N_1/2$) of the points y_i ($i \in J_1$) lie outside $2\tau B(z_1)$; so there is $I_2 = \{i \in J_1 : y_i \notin 2\tau B(z_1)\}$ with $\#I_2 \geq N_1/2$. Put $d_1 = d(z_1)$ and let x_{2i} be the first point along $[x_{1i}, y_i]$ with $|x_{2i} - z_1| = 2\tau d_1$. Then

$$(1/2)d_1 \leq (1-2\tau)d_1 \leq d(x_{2i}) \leq (1+2\tau)d_1 \leq (3/2)d_1,$$

and so

$$k(x_{1i}, x_{2i}) \geq \tau/2 \qquad \text{for all } i \in I_2.$$

Again, one of the points x_{2i} enjoys the property that $\tau B(x_{2i})$ contains at least $N_1/(2M^2 24^Q)$ of the points x_{2j} ($j \in I_2$). Thus, there exists a point z_2 (one on the x_{2i}, $i \in I_2$) and $J_2 \subset I_2$ such that

$$x_{2j} \in \tau B(z_2) \quad \text{for all } j \in J_2, \text{ and } \quad \#J_2 \geq N_2 = N/(2M^2 24^Q)^2.$$

Suppose we have followed this procedure and obtained points $z_1, z_2, \ldots, z_n$ (where z_k is some point x_{ki}, $i \in I_k$) and index sets $J_n \subset I_n \subset \cdots \subset J_1 \subset I_1$ such that for all $1 \leq k \leq n$,

$$x_{kj} \in \tau B(z_k) \quad \text{for all } j \in J_k, \text{ and } \quad \#J_k \geq N_k = N/(2M^2 24^Q)^k.$$

If at least $N_n/2$ of the points y_i ($i \in J_n$) lie outside $2\tau B(z_n)$, we can continue as above to obtain our next 'special' point z_{n+1}. However, if $n > 2T/\tau$, and say $z_n = x_{nj}$, then

$$k(z_0, y_j) \geq k(z_0, z_n) = k(z_0, x_{1j}) + \sum_{i=2}^{n} k(x_{i-1,j}, x_{ij}) \geq \tau + (n-1)\tau/2 = n\tau/2 > T,$$

which violates the hypothesis that $\Sigma \subset B_k(z_0; T)$.

We conclude that the above process must stop at some time $n \leq 2T/\tau$. This means at least $N_n/2$ of the points y_i ($i \in J_n$) must lie inside the ball $2\tau B(z_n)$. Returning once more to the proof of (3.7) we conclude that $N_n \leq 2M^2 8^Q$, so

$$N = N_n(2M^2 24^Q)^n \leq 2M^2 8^Q (2M^2 24^Q)^{2T/\tau} = 2M^2 8^Q (2M^2 24^Q)^{8aT/\lambda t}$$

as desired. □

A standard argument shows that the above local doubling condition guarantees that our space has bounded growth (at all scales) in the sense that for all $R > r > 0$ there is an $N = N(R/r)$ such that each ball of radius R in (Ω, k) can be covered using at most N balls of radius r. Employing a result of Bonk and Schramm [BS00, Thm.9.2] we obtain the following consequence of these ideas.

3.8. **Corollary.** *Let Ω be a locally quasiconvex locally Ahlfors regular abstract domain with a Gromov hyperbolic quasihyperbolization. Then $\partial_G\Omega$ is a doubling metric space.*

3.E. **Proof of Theorem C.** As discussed at the beginning of this Section, we may assume that Ω is a bounded a-uniform locally Ahlfors Q-regular locally Q-Loewner space. Suppose also that $\partial_G\Omega$ is uniformly perfect, and that ν is a P-dimensional metric doubling measure on $\partial_G\Omega$ with $P < Q$. (Since the conformal gauge determined by d on $\partial\Omega$ is naturally QS equivalent to the canonical gauge on $\partial_G\Omega$, we see that $\partial\Omega$ is uniformly perfect too and that ν is a P-dimensional metric doubling measure on $\partial\Omega$. Here and below we identify $\partial_G\Omega$ with $\partial\Omega$ via this natural quasisymmetric map explained in Fact 2.14.) Recall that this means we can use ν to define a quasimetric $\delta = \delta_{\nu,1/P}$ on $\partial_G\Omega \simeq \partial\Omega$ by

$$\delta(\xi,\eta) := \nu[B(\xi\eta)]^{1/P}, \quad \text{where } B(\xi\eta) := \partial\Omega \cap \big(\bar{B}(\xi;r) \cup \bar{B}(\eta;r)\big) \text{ and } r = |x-y|.$$

Also: δ is bilipschitz equivalent to a distance function on $\partial\Omega$, the identity map $(\partial\Omega,\delta) \to (\partial\Omega,d)$ is quasisymmetric and $(\partial\Omega,\delta,\nu)$ is Ahlfors P-regular. See Fact 4.9 and the comments immediately following.

We define the *lift* $\rho = \rho_\nu$ of ν via the formula

$$\rho(x) := \frac{\nu(S_x)^{1/P}}{d(x)} \qquad \text{where } S_x := \partial\Omega \cap 2B(x). \tag{3.9}$$

We verify that ρ is a Borel Harnack doubling Ahlfors Koebe metric-density on Ω. ($\ddot\smile$)

Choose a point $w \in \Omega$ with $d(w)$ maximal; see Lemma 2.12(a). Notice that if $\xi \in \partial\Omega$ satisfies $d(x) = |x-\xi|$, then

$$\partial\Omega \cap B(\xi;d(x)) \subset S_x \subset \partial\Omega \cap B(\xi;3d(x)). \tag{3.10}$$

We claim also that for any $x \in \Omega$ and $\zeta \in \partial\Omega \cap bB(x)$, $k(x,[w,\zeta)) \le C(a,b)$. Indeed, since Ω is a bounded a-uniform space with 'center' w, Lemma 2.12(a) asserts that $|z-\zeta| \le 4a^2 d(z)$ for each $z \in [w,\zeta)$. Choosing $z \in [w,\zeta)$ with $d(z) = d(x)$ we have $|x-z| \le (b+4a^2)d(x)$ and so Fact 2.11 permits us to deduce that $k(x,z) \le 4a\log(1+b+4a^2)$. (Of course if $\zeta \in S_x$, then $b = 2$).

Proof that ρ is Borel and Harnack. It is straightforward to show that $x \mapsto \nu(S_x)$ is upper-semicontinuous, so ρ is too. (This is the only place were we need the flexibility of allowing for Borel densities as opposed to requiring continuity. We could obtain the latter if we knew that ν was absolutely continuous in an appropriate sense.) That ρ is Harnack is a consequence of the doubling property of ν and (3.10); in fact, we can use the Whitney ball constant $\lambda = 1/2$ (for ρ) and we find that H_ρ depends only on Q and the doubling constant for ν. $\diamond$

Knowing that ρ is Harnack, we readily establish the following information.

$$\operatorname{diam}_\delta S_z \simeq \nu(S_z)^{1/P} \simeq \mu_\rho[\sigma B(z)]^{1/Q} \simeq \operatorname{diam}_\rho[\sigma B(z)]. \tag{3.11}$$

Here and below we fix $\sigma = \lambda/(10a)$. The leftmost inequalities above follow from the fact that $(\partial\Omega,\delta,\nu)$ is Ahlfors P-regular in conjunction with (3.10); the constant depends only on P, the doubling constant for ν, and the uniformly perfect constant for $\partial\Omega$. The middle and rightmost inequalities hold because ρ is Harnack and μ is locally Ahlfors Q-regular; the constants depend only on σ, H_ρ, M_μ, Q (see also Lemma 2.2).

Proof that ρ is doubling. We must verify that μ_ρ is a doubling measure on (Ω,d). Let $x \in \Omega$ and $r > 0$. Since ρ is Harnack and μ is locally Ahlfors Q-regular, we easily obtain the doubling condition with constant $M_\mu^2(2H_\rho^2)^Q$ if $r \le (\lambda/2)d(x)$. Thus we assume that $r > (\lambda/2)d(x)$. According to Lemma 2.12(b), there is a point $z \in \Omega$ with

$$(\lambda/5)B(z) \subset B(x;r) \qquad \text{and} \qquad B(x;2r) \subset (9a/\lambda)B(z).$$

Thus we see that our task will be complete once we demonstrate that

$$\mu_\rho[\sigma^{-1}B(x)] \lesssim \mu_\rho[\sigma B(x)] \qquad \text{for every } x \in \Omega,$$

where the constant involved in the inequality depends only on the data associated with Ω.

Let $\mathcal{C}$ be a maximal collection of points in $\Omega \cap \sigma^{-1}B(x)$ satisfying $k(y,z) \ge \log(1+\sigma)$ for all points $y \ne z$ in $\mathcal{C}$. Using the basic estimates for k, it is easy to check that $\mathcal{C}$ being maximal ensures that,

$$\Omega \cap \sigma^{-1}B(x) \subset \bigcup_{z\in\mathcal{C}} \sigma B(z).$$

For $m = 0,1,2,\dots$ let $\mathcal{C}_m := \{z \in \mathcal{C} : m \le k(z,x) < m+1\}$. Then $\mathcal{C}$ is a disjoint union of the sets $\mathcal{C}_m$. We claim that for m fixed, $k(y,z) \le C(a,\lambda)$ for all $y,z \in \mathcal{C}_m$ with $S_z \cap S_y \ne \emptyset$. Indeed, fix $y,z \in \mathcal{C}_m$. First note that if $d(y) \ge d(x)$, then Fact 2.11 gives us $m \le C$. Thus if either $d(y) \ge d(x)$ or $d(z) \ge d(x)$, then $k(y,z) \le 2(m+1) \le C$. Thus we may assume that both $d(y) < d(x)$ and $d(z) < d(x)$. Suppose too that $S_z \cap S_y \ne \emptyset$, and let ζ be a boundary point in both shadows. Let x', y', z' be the last points on $[w,\zeta)$ with $d(x') = d(x)$, $d(y') = d(y)$, $d(z') = d(z)$ and note that x' cannot lie between y' and z' (so $k(y',z') = |k(x',z') - k(x',y')|$). Since $\zeta \in S_y \cap S_z$, we have $k(y,y')$ and $k(z,z')$ both bounded by $4a\log(3+4a^3)$ (see the paragraph following (3.10)); since $|\zeta - x| \le (2+3\sigma^{-1})d(x)$, $k(x,x') \le C$. Several applications of the triangle inequality yield

$$k(y,z) \le k(y,y') + k(y',z') + k(z',z) \le C(a,\lambda).$$

It now follows from Proposition 3.6 that for each fixed m, the shadows S_z for $z \in \mathcal{C}_m$ have uniformly bounded overlap; that is,

$$\sum_{z\in\mathcal{C}_m} \chi_{S_z} \le C = C(a,\lambda,M,Q).$$

Next, Fact 2.11 provides constants $0 < \vartheta = e^{-1/4a} < 1$ and $C = C(a,\lambda) \ge 1$ such that for each $m = 0,1,2,\dots$

$$d(z) \le C\vartheta^m d(x) \qquad \text{for all } z \in \mathcal{C}_m.$$

Finally, we find constants $C \ge 1$, $\alpha > 0$ depending only on a, λ, the doubling constant for ν, and the uniform perfectness constant with

$$\nu(S_z) \le C\vartheta^{\alpha m}\nu(S_x) \qquad \text{for all } z \in \mathcal{C}_m.$$

(Apply Fact 4.7 using (3.10) along with $B(\zeta;3d(z)) \subset B(\xi;5(1+\sigma^{-1})d(x))$, where ξ,ζ are boundary points with $d(x) = |x-\xi|$, $d(z) = |z-\zeta|$.)

Thus, using bounded overlap of the shadows S_z (and the fact from above that $S_z \subset \partial\Omega \cap B(\xi;5(1+\sigma^{-1})d(x))$) for $z \in \mathcal{C}_m$ and each $m = 0,1,2,\dots$, we deduce that

$$\sum_{z\in\mathcal{C}_m} \nu(S_z)^{Q/P} \le \max_{y\in\mathcal{C}_m} \nu(S_y)^{(Q/P)-1} \sum_{z\in\mathcal{C}_m} \nu(S_z) \le C\vartheta^{\alpha[(Q/P)-1]m}\nu(S_x)^{Q/P}.$$

Since this holds for all $m = 0, 1, 2, \dots,$

$$\mu_\rho[\sigma^{-1}B(x)] \le \sum_{z\in\mathcal{C}} \mu_\rho[\sigma B(z)] \simeq \sum_{z\in\mathcal{C}} \nu(S_z)^{Q/P} =$$

$$= \sum_{m=0}^{\infty}\sum_{z\in\mathcal{C}_m} \nu(S_z)^{Q/P} \lesssim \sum_{m=0}^{\infty} \vartheta^{\alpha[(Q/P)-1]m}\nu(S_x)^{Q/P} \simeq \mu_\rho[\sigma B(x)]$$

as desired. $\diamond$

Proof that ρ is Ahlfors. Let $x \in \Omega$ and $r > 0$. We must verify that ρ satisfies the volume growth condition (A). Since μ is locally Ahlfors Q-regular and ρ is Harnack, Corollary 2.3 says that (A) holds when $r \le (\lambda/2a^2H)\rho(x)d(x)$; assume $r \ge (\lambda/2a^2H)\rho(x)d(x)$. Using Lemma 2.2 we find that $\operatorname{diam} B_\rho(x;r) \ge 2c\, d(x)$ where $c = \lambda/4a^3H^2$. Thus we may choose a point $y \in B_\rho(x;r)$ and an arc γ joining x to y so that

$$|x-y| \ge (1/2)\operatorname{diam} B_\rho(x;r) \ge c\, d(x) \qquad \text{and} \qquad d_\rho(x,y) \le \ell_\rho(\gamma) \le r. \tag{3.12}$$

We claim that for each point $z \in \gamma$,

$$\operatorname{diam}_\rho[\sigma B(z)] \lesssim \ell_\rho(\gamma \cap \sigma B(z)). \tag{3.13}$$

To see this, fix $z \in \gamma$ and recall from Lemma 2.2 that $\operatorname{diam}_\rho[\sigma B(z)] \simeq \rho(z)d(z)$. If $\gamma \subset \sigma B(z)$, then

$$\log(1+c) \le k(x,y) \le \ell_k(\gamma) \le \frac{H}{1-\sigma}\frac{\ell_\rho(\gamma)}{\rho(z)d(z)};$$

otherwise γ leaves $\sigma B(z)$ in which case the estimate follows from ρ being Harnack.

Now we consider two cases: first, suppose that there is a $z \in \gamma$ with $d(z) \gtrsim \operatorname{diam} B_\rho(x;r)$; so $B_\rho(x;r) \subset C\, B(z)$ for some constant C. Then since ρ is doubling and Harnack, and μ is locally Ahlfors Q-regular, we can use (3.11) to obtain

$$\mu_\rho[B_\rho(x;r)] \lesssim \mu_\rho[\sigma B(z)] \simeq \operatorname{diam}_\rho[\sigma B(z)]^Q \lesssim \ell_\rho(\gamma)^Q \le r^Q,$$

where the penultimate inequality follows from (3.13). Thus (A) holds as desired.

We are left to handle the case when there is no such point z; so γ stays fairly close to $\partial\Omega$. In particular, we now have, say, $d(x) \le (1/3)|x-y|$ and $d(y) \le (1/3)|x-y|$. Select points $\xi, \eta \in \partial\Omega$ with $d(x) = |x-\xi|$, $d(y) = |y-\eta|$. Then $|\xi-\eta| \ge (1/3)|x-y|$. Let z be the arclength midpoint of $[\xi,\eta]$. Then

$$d(z) \ge (1/2a)\ell[\xi,\eta] \ge \frac{|\xi-\eta|}{2a} \ge \frac{|x-y|}{6a} \ge (1/12a)\operatorname{diam} B_\rho(x;r).$$

Thus $B_\rho(x;r) \subset 12aB(z)$. We show below that

$$\nu(S_z)^{1/P} \lesssim \delta(\xi,\eta) \lesssim \ell_\rho(\gamma),$$

and therefore using doubling for ρ, (3.11), and (3.12) we conclude that

$$\mu_\rho[B_\rho(x;r)] \lesssim \mu_\rho[\sigma B(z)] \simeq \nu(S_z)^{Q/P} \lesssim r^Q.$$

Recall that $\delta(\xi,\eta) = \nu[B(\xi\eta)]^{1/P}$, where $B(\xi\eta) := \partial\Omega \cap [\bar{B}(\xi;R) \cup \bar{B}(\eta;R)]$ and $R = |\xi-\eta|$. It is easy to see that $S_z \subset B(\xi; 2aR)$, so $\nu(S_z)^{1/P} \lesssim \delta(\xi,\eta)$ is an easy consequence of the doubling property for ν. Thus it remains to establish that $\delta(\xi,\eta) \lesssim \ell_\rho(\gamma)$.

Notice that for any points $u, v \in \Omega$ and $\zeta \in \partial\Omega$: if $v \in (1/2)B(u)$ and $d(v) = |v-\zeta|$, then $\zeta \in S_u$. Thus in particular, if $5\sigma B(z) \cap 5\sigma B(w) \ne \emptyset$, then $S_z \cap S_w \ne \emptyset$. Now consider the collection $\{\sigma B(z) : z \in \gamma\}$ of balls which cover γ. Appealing to a Basic Covering Theorem

[Hei01, Thm.1.2], we get pairwise disjoint balls $B_i = \sigma B(z_i)$ ($z_i \in \gamma$) such that γ is covered by the balls $5B_i = 5\sigma B(z_i)$. Since γ is compact, we can further select a finite subcover, and then by relabeling (if necessary) we may assume that there are distinct balls $B_0, B_1, \dots, B_m$ such that $x \in 5B_0$, $y \in 5B_m$, and $5B_i \cap 5B_{i-1} \neq \emptyset$ for $1 \le i \le m$.

Now $5B_i \cap 5B_{i-1} \neq \emptyset$ ensures that $S_i \cap S_{i-1} \neq \emptyset$ where $S_i = S_{z_i}$. Thus $S_0, S_1, \dots, S_m$ is a chain of overlapping sets joining ξ to η. Therefore using (3.11) and (3.13) we deduce that

$$\delta(\xi,\eta) \le \sum_0^m \operatorname{diam}_\delta(S_i) \simeq \sum_0^m \operatorname{diam}_\rho(B_i) \le \sum_0^m \ell_\rho(\gamma \cap B_i) \le \ell_\rho(\gamma),$$

where the last inequality holds because the B_i are pairwise disjoint. $\diamond$

Proof that ρ is Koebe. Since $\partial\Omega$ is uniformly perfect, it enjoys a Hausdorff measure density condition; see Lemma 4.2. An appeal to [Her04, Thm.F] permits us to assert that Ω satisfies a Whitney ball modulus condition. (Note that Ω is locally Q-Loewner by Corollary 3.3 and also both Ω and $\partial\Omega$ are bounded.) That ρ is a Koebe density now follows from [Her04, Thm.E], since we already know that ρ is both Harnack and Ahlfors. $\diamond$

Finally, it remains to certify that the natural map $(\partial_\rho\Omega, d_\rho) \to (\partial_G\Omega, \delta)$ is bilipschitz. Notice that here we have $\partial_G\Omega \simeq \partial\Omega = \partial_\rho\Omega$ as sets, the latter equality being due to Proposition 3.4 (since ρ is doubling and Ω is both bounded and locally Q-Loewner).

Fix points $\xi, \eta \in \partial\Omega$ and let z be the arclength midpoint of $[\xi,\eta]$. Using the fact that $|\xi - \eta| \simeq d(z)$ it is not hard to see that S_z and $B(\xi\eta)$ are comparable (cf. (3.10)), and thus according to (3.11) and (3.9)

$$\delta(\xi,\eta) \simeq \nu(S_z)^{1/P} = \rho(z)d(z).$$

To estimate $d_\rho(\xi,\eta) \simeq \ell_\rho[\xi,\eta]$ we examine the penultimate paragraph of the proof of Proposition 3.4 to remember that this distance is comparable to $\rho(x)d(x)$ where $x \in [\xi,\eta]$ is the furthest point from $\partial\Omega$. Then $d(z) \le d(x) \le 2a^2 d(z)$. Also, $|x - z| \le a^2 d(z)$, so $k(x,z) \le 4a\log(1+a^2)$; therefore by Lemma 2.8 we deduce that $\rho(x) \simeq \rho(z)$. We conclude that

$$d_\rho(\xi,\eta) \simeq \rho(x)d(x) \simeq \rho(z)d(z) \simeq \delta(\xi,\eta)$$

as desired. $\square$

3.14. **Remark.** The reader should not misinterpret our definition of the lift of a (metric) doubling measure ν on $\partial_G\Omega$; (3.9) is valid only for the BHK-uniformized space. Certainly one way to lift ν to the 'original' abstract domain Ω is to pull-back, via the uniformizing density π_ε, the lift given by (3.9). A more natural, and more direct, definition is to define the *lift* $\rho = \rho_\nu$ of ν via

$$\rho(x) := \frac{\nu(\Sigma_x)^{1/P}}{d(x)}; \tag{3.15}$$

here ν is a P-dimensional metric doubling measure on $\partial_G\Omega$ and, for some fixed base point $w \in \Omega$, the *shadow* of a point $x \in \Omega$ is

$$\Sigma_x := \{\zeta \in \partial_G\Omega : k(x,[w,\zeta)) \le R\}.$$

It is not hard to see that there are constants $R = R(\Delta,\kappa,\varepsilon)$ and $C = C(\Delta,\kappa,\varepsilon)$ such that

$$S_x := \partial_\varepsilon\Omega \cap 2B_\varepsilon(x) \subset \Sigma_x \subset \partial_\varepsilon\Omega \cap CB_\varepsilon(x); \tag{3.16}$$

of course we are using the natural identification of $\partial_G\Omega$ with $\partial_\varepsilon\Omega$. Employing (3.16), the doubling property of ν, and the fact that π_ε is Koebe, it is straightforward to verify that the push-forward, via the uniformizing density π_ε, of ρ_ν as defined in (3.15) gives a density on Ω_ε which is bilipschitz equivalent to the density defined by (3.9).

Let us indicate why (3.16) is true. Here we assume that (Ω, d) is a locally a-quasiconvex abstract domain having a Gromov hyperbolic roughly starlike quasihyperbolization (Ω, k). Also, $\Omega_\varepsilon = (\Omega, d_\varepsilon)$ is the BHK-uniformization of (Ω, d) via the metric-density $\pi_\varepsilon(x) = \rho_\varepsilon(x)/d(x) = d(x)^{-1}\exp(-\varepsilon k(x, w))$ with $w \in \Omega$ a fixed base point and $0 < \varepsilon < \varepsilon(\Delta) = (300 \max\{1, \Delta\})^{-2}$; see §2.F and especially Lemma 2.17. We remind the reader that quasihyperbolic geodesics in Ω are 20-uniform arcs in Ω_ε, so the quasihyperbolic geodesics in Ω_ε are b-uniform arcs in Ω_ε where b is an absolute constant (e.g., $b = \exp(1000 \cdot 20^6)$ works, according to [BHK01, Thm.2.10]).

Mimicking the proof of Lemma 2.12(a) we find that for all $\zeta \in \partial_\varepsilon\Omega$ and all $z \in [w, \zeta)$,

$$d_\varepsilon(z) \le d_\varepsilon(z, \zeta) \le 80b d_\varepsilon(z).$$

Fix $x \in \Omega$. We produce constants R and C so that (3.16) holds.

First, start with a point $\zeta \in S_x := \partial_\varepsilon\Omega \cap 2B_\varepsilon(x)$. From the paragraph immediately following (3.10) we deduce that

$$k_\varepsilon(x, [w, \zeta)_\varepsilon) \le k_1 \qquad \text{where } k_1 \text{ is an absolute constant.}$$

Select points $y \in [w, \zeta)_\varepsilon$ and $z \in [w, \zeta)$ with

$$k_\varepsilon(x, y) = k_1 \quad \text{and} \quad d_\varepsilon(z) = d_\varepsilon(y).$$

Using Lemma 2.12(a) and the above remark, we obtain

$$d_\varepsilon(y, \zeta) \le 4b^2 d_\varepsilon(y) \quad \text{and} \quad d_\varepsilon(z, \zeta) \le 80b d_\varepsilon(z).$$

Therefore, $d_\varepsilon(y, z) \le 4b(b + 20)d_\varepsilon(y)$, so by Fact 2.11, $k_\varepsilon(y, z) \le k_2$ where k_2 is an absolute constant. Thus,

$$k(x, [w, \zeta)) \le k(x, z) \le K k_\varepsilon(x, z) \le K\,(k_\varepsilon(x, y) + k_\varepsilon(y, z)) \le (k_1 + k_2)K;$$

here $K = \max\{\varepsilon e, (2e^{\varepsilon\kappa} - 1)/\varepsilon\}$. Hence $R = (k_1 + k_2)K$ 'works'.

Next, let $\zeta \in \Sigma_x := \{\zeta \in \partial_G\Omega : k(x, [w, \zeta)) \le R\}$. Select $y \in [w, \zeta)$ with $k(x, y) = k(x, [w, \zeta))$; then $k_\varepsilon(x, [w, \zeta)) \le k_\varepsilon(x, y) \le KR$, so

$$d_\varepsilon(x, y) \le \left(e^{KR} - 1\right)(d_\varepsilon(x) \wedge d_\varepsilon(y)) \quad \text{and} \quad e^{-KR} \le \frac{d_\varepsilon(x)}{d_\varepsilon(y)} \le e^{KR}.$$

Since $d_\varepsilon(y, \zeta) \le 80b d_\varepsilon(y) \le 80be^{KR} d_\varepsilon(x)$, we obtain

$$d_\varepsilon(x, \zeta) \le d_\varepsilon(x, y) + d_\varepsilon(y, \zeta) \le e^{KR}(80b + e^{KR})d_\varepsilon(x)$$

and therefore $C = e^{KR}(80b + e^{KR})$ 'works'.

3.F. **Proof of Theorem A.** Let (Ω, d, μ) be an abstract domain with bounded Q-geometry. Suppose that ρ is a bounded uniformizing conformal density on Ω. Thus $\Omega_\rho = (\Omega, d_\rho)$ is a bounded a-uniform space, so by [BHK01, Thm.3.6], its quasihyperbolization, (Ω_ρ, k_ρ), is a proper geodesic Gromov Δ_ρ-hyperbolic roughly κ_ρ-starlike space with Δ_ρ and κ_ρ depending only on a. Since ρ is Koebe, the identity map $(\Omega, k) \to (\Omega_\rho, k_\rho)$ is K_ρ-bilipschitz. Therefore, (Ω, k) is also proper, geodesic, Gromov Δ-hyperbolic, and roughly κ-starlike space where Δ and κ depend only on K_ρ and a. (The fact that bilipschitz homeomorphisms preserve Gromov hyperbolicity is a consequence of the geodesic stability property for such spaces;

see [Bon96, Prop.3.1], [BH99, Thm.III.H.1.7], [BBI01, Cor.8.4.17, Thm.8.4.20]. That rough starlikeness is also preserved can be verified in a similar manner.)

It remains to establish the dimension estimate for the Gromov boundary of Ω. To this end, note that we have quasisymmetric equivalences

$$\partial_G\Omega \simeq \partial_G\Omega_\rho \simeq \partial_\rho\Omega;$$

the first of these holds because the identity map $(\Omega, k) \to (\Omega_\rho, k_\rho)$ is bilipschitz (cf. [BS00, Thm.6.5(2)]) while the second equivalence is part of [BHK01, Thm.3.6 (cf. pp.24-25)]. Next, an appeal to Theorem B permits us to assert that Ω_ρ is Ahlfors Q-regular with a constant which depends only on the data associated with Ω and with ρ. By extending μ_ρ to be zero on the boundary $\partial_\rho\Omega$ we find that $\bar{\Omega}_\rho$ is also Ahlfors Q-regular. It is easy to see that Ω_ρ uniform implies that $\partial_\rho\Omega$ is porous in $\bar{\Omega}_\rho$. According to Fact 4.5 this means that $\dim_{\mathcal{A}}\partial_\rho\Omega \le Q - c$ where $c > 0$ depends only on Q, the uniformity constant, and the Hausdorff Q-regularity constant for Ω_ρ. Since $\partial_G\Omega \simeq \partial_\rho\Omega$ we obtain c-$\dim_{\mathcal{A}}\partial_G\Omega \le Q - c$ as desired.

Let us now substantiate the converse. As discussed at the beginning of this Section, we may assume that Ω is a bounded uniform locally Ahlfors Q-regular locally Q-Loewner space. In addition we suppose that the Gromov boundary of Ω is uniformly perfect and has conformal Assouad dimension strictly less than Q. Let ∂ be a distance in the canonical conformal gauge on $\partial_G\Omega$ with $\dim_{\mathcal{A}}(\partial_G\Omega, \partial) < Q$ and fix P with $\dim_{\mathcal{A}}(\partial_G\Omega, \partial) < P < Q$. According to Fact 4.6, there is a P-homogeneous measure ν on $(\partial_G\Omega, \partial)$. Since ν is then a P-dimensional metric doubling measure on $\partial_G\Omega$, Theorem C tells us that its lift, ρ_ν, is a doubling conformal density on Ω. Then according to Theorem B, ρ_ν is also a bounded uniformizing density on Ω. □

3.G. **Proof of Theorem D.** As discussed at the beginning of this Section, we may assume that Ω is a bounded uniform locally Ahlfors Q-regular locally Q-Loewner space. (We should perhaps say a bit more here. It is easy to see that conformal doubling densities on Ω correspond to the same on Ω_ε, and of course the deformed spaces are identical. We must also consider Remark 3.14.) Suppose also that $\partial_G\Omega$ is uniformly perfect and fix $P < Q$. Recall that $\partial_G\Omega$ is naturally QS equivalent to $\partial\Omega$ (cf. Fact 2.14), so $\partial\Omega$ is uniformly perfect too.

Let ρ be a metric-density on Ω. Suppose ρ is induced by some P-dimensional metric doubling measure ν on $\partial_G\Omega$. In particular then, $(\partial_G\Omega, \delta_\nu, \nu)$ is Ahlfors P-regular, where $\delta_\nu = \delta_{\nu,1/P}$ is the quasimetric induced by ν; see (4.8) and Fact 4.9.

According to Theorem C, the lift ρ_ν of ν is a doubling conformal density on Ω. Since $\rho \simeq \rho_\nu$, ρ is also a doubling conformal density on Ω. Hence by Proposition 3.4 we have $\partial_\rho\Omega = \partial\Omega = \partial_{\rho_\nu}\Omega$ (as sets). Thus we have bilipschitz maps

$$(\partial_\rho\Omega, d_\rho) \to (\partial_{\rho_\nu}\Omega, d_{\rho_\nu}) \to (\partial_G\Omega, \delta_\nu);$$

the first map above is the identity (and is bilipschitz because $\rho \simeq \rho_\nu$) while the second is the natural map as in Theorem C. Since $(\partial_G\Omega, \delta_\nu)$ is Ahlfors P-regular, so is $(\partial_\rho\Omega, d_\rho)$.

Now assume that ρ is known to be a doubling conformal density on Ω with $(\partial_\rho\Omega, d_\rho)$ Ahlfors P-regular. This means that the Hausdorff P-dimensional measure $\mathcal{H}^P_\rho$ on $\partial_\rho\Omega$ is Ahlfors P-regular, and the quasimetric induced by $\mathcal{H}^P_\rho$ on $\partial_\rho\Omega$ is bilipschitz equivalent to d_ρ. By Proposition 3.4, $\partial_\rho\Omega = \partial\Omega$ (as sets). Appealing to Theorem B we deduce that the identity map $\Omega_\rho \to \Omega$ is quasisymmetric; this extends to their closures and boundaries also.

Since $\partial_G\Omega$ is naturally QS equivalent to $\partial\Omega$, we see that the pull-back, call it ν, of $\mathcal{H}^P_\rho$ to $\partial_G\Omega$ is a P-dimensional metric doubling measure on $\partial_G\Omega$. It remains to demonstrate that $\rho \simeq \rho_\nu$, where the lift ρ_ν of ν is given by (3.9). Let $x \in \Omega$ and choose a point $\xi \in \partial\Omega$ with $d(x) = |x-\xi|$. By quasisymmetry we find that $d_\rho(x) \simeq d_\rho(x,\xi)$ and therefore that the ball $B(\xi; d(x))$ is comparable to $B_\rho(\xi; d_\rho(x))$. Appealing to (3.10) we deduce that

$$\rho_\nu(x)d(x) = \nu(S_x)^{1/P} \simeq \ \mathcal{H}^P_\rho[B_\rho(\xi; d_\rho(x))]^{1/P} \simeq d_\rho(x) \simeq \rho(x)d(x),$$

where the penultimate $\simeq$ holds because $\mathcal{H}^P_\rho$ is Ahlfors P-regular on $\partial_\rho\Omega$. □

4. Metric Space Miscellanea

Here we collect standard information for general metric measure spaces (X, d, μ).

4.A. **Geodesics, Quasimetrics, Dimensions, and Uniform Perfectness.** Recall that a *geodesic* in a metric space X is the image $\varphi(I)$ of some isometric embedding $\varphi : I \to X$ where $I \subset \mathbf{R}$ is an interval; we use the adjectives *segment, ray,* or *line* (respectively) to indicate that I is bounded, semi-infinite, or all of $\mathbf{R}$. If φ is L-bilipschitz, then we call $\varphi(I)$ an *L-quasigeodesic*. A metric space is *geodesic* if each pair of points can be joined by a geodesic segment. We use the notation $[x, y]$ to mean a (not necessarily unique) geodesic segment (or ray or line) joining points x, y; such geodesics always exist if our space is geodesic, but may not be unique. We consider a given geodesic $[x, y]$ as being ordered from x to y (so we can use phrases such as the 'first' point encountered). An unbounded metric space is *roughly κ-starlike*, with respect to a base point w, if each point lies within distance κ of some geodesic ray emanating from w.

An embedding $f : X \to Y$ of a metric space X into a metric space Y is termed *quasisymmetric*, abbreviated QS, if there is a homeomorphism $\eta : [0,\infty) \to [0,\infty)$ (called a *distortion function*) such that for all triples $x, y, z \in X$,

$$|x-y| \le t|x-z| \implies |fx-fy| \le \eta(t)|fx-fz|.$$

These mappings were studied by Tukia and Väisälä in [TV80]; see also [Hei01]. The *bilipschitz* maps form an important subclass of the quasisymmetric maps; $f : X \to Y$ is bilipschitz if there is a constant L such that for all $x, y \in X$,

$$L^{-1}|x-y| \le |fx-fy| \le L|x-y|.$$

The above definitions also make sense for mappings of quasimetric spaces. We call q a *quasimetric* on X if q is symmetric and positive definite on $X \times X$ but only satisfies

$$q(x,y) \le K\,(q(x,z)+q(y,z)) \qquad \text{for all } x,y,z \in X$$

in place of the triangle inequality. See [Hei01, 14.1]. There is a standard way to use a quasimetric q to define a pseudometric d with $d \le q$ (cf. [BH99, 1.24,p.14]), but it may happen that $d(x,y) = 0$ for some $x \ne y$. However, by first 'snowflaking' q and then applying this procedure we can arrive at an honest distance function; see [Hei01, Prop.14.5] or [BH99, Prop.3.21,p.435].

4.1. **Fact.** *Let q be a quasimetric on X. There is an $\varepsilon_0 > 0$, depending only on the quasimetric constant K for q, such that for all $\varepsilon \in (0, \varepsilon_0)$, the quasimetric $q_\varepsilon(x,y) = q(x,y)^\varepsilon$ is bilipschitz equivalent to an honest distance function d on X; in fact there is a constant $L = L(\varepsilon, K)$ such that*

$$L^{-1}q_\varepsilon(x,y) \le d(x,y) \le q_\varepsilon(x,y) \qquad \textit{for all } x,y \in X.$$

We remark that all the quasimetrics q_ε as defined above are QS equivalent to each other. Now, given a metric (or a quasimetric) on X, we can form the *conformal gauge* on X consisting of all metrics on X which are QS equivalent to the original (quasi)metric. See [Hei01, Chpt.15] for more discussion of this topic.

Recall that *α-dimensional Hausdorff measure* is given by

$$\mathcal{H}^\alpha(A) = \liminf_{r\to 0}\{\sum \operatorname{diam}(B_i)^\alpha : A \subset \cup B_i, B_i \text{ balls with } \operatorname{diam}(B_i) \le r\}$$

and the *Hausdorff dimension* of A is determined by

$$\dim_{\mathcal{H}}(A) = \inf\left\{\alpha > 0 : \mathcal{H}^\alpha(A) = 0\right\}.$$

We also require the *Assouad dimension* of X which is given by

$$\dim_{\mathcal{A}}(X) = \inf\{\alpha : \#S \le C(R/r)^\alpha \text{ for all } S \subset X \text{ with } r \le |x-y| \le R \text{ for all } x, y \in S\}$$

where $\#S$ denotes the cardinality of the set S. See [Hei01, 10.15], [Luu98, 3.2]. The spaces with finite Assouad dimension are precisely the doubling spaces (which we discuss in more detail in subsection 4.E). Finally, the *conformal Assouad dimension* of a metric space X is

$$\text{c-}\dim_{\mathcal{A}}(X) = \inf\{\dim_{\mathcal{A}}(X, d) : d \in \mathcal{G}\},$$

where $\mathcal{G}$ is the conformal gauge on X determined by the original metric; see [Hei01, 15.8,p.125].

A metric space X is called *uniformly perfect* provided there is a constant $0 < \vartheta < 1$ such that for all balls $B \subset X$, $B \setminus \vartheta B \ne \emptyset$ provided $X \setminus B \ne \emptyset$. This concept, which involves three points, is especially useful when dealing with quasisymmetric maps and also with doubling measures (see subsection 4.E). The property of being uniformly perfect is preserved by quasisymmetric homeomorphisms, with the new constant depending only on the original constant and the quasisymmetry data; in particular, one can ask whether or not a conformal gauge is uniformly perfect.

It is a routine exercise to see that uniformly perfect locally compact spaces contain quasisymmetrically embedded middle-third Cantor dusts. Using this fact, together with a scaling argument and properties of quasisymmetric homeomorphisms (e.g. [Hei01, 11.10,11.11], one can verify a version of the following. We prefer a simpler more direct approach which also provides explicit constants; for the reader's convenience we outline the argument.

4.2. **Lemma.** *Suppose X is a uniformly perfect compact metric space. Then X satisfies the α-dimensional Hausdorff measure density condition*

$$\mathcal{H}^\alpha[B(x;r)] \ge \frac{r^\alpha}{6} \quad \textit{for all } 0 < r \le \operatorname{diam}(X) \textit{ and all } x \in X,$$

where $\alpha = \log 2/\log(4/\vartheta) \in (0, 1/2)$ and ϑ is the uniform perfectedness constant.

Proof. Fix points x_0, x_1 in X with $|x_0 - x_1| \ge R = \operatorname{diam}(X)/2$. We get points $x_{11} = x_0, x_{12}, x_{13}, x_{14} = x_1$ with $x_{12} \in (3/4)A_{11} \setminus (3\vartheta/4)A_{11}$, $A_{11} = \bar{B}(x_{11}; R_1)$, $R_1 = (\vartheta/4)R$ and similarly for x_{13}. Continuing this construction for $i = 1, 2, \cdots$ we obtain 'nested' closed balls $A_{ij} := \bar{B}(x_{ij}; R_i)$ $(1 \le j \le 2^{i+1})$ where $R_i := (\vartheta/4)^i R$, $x_{i+1,2j-1} = x_{ij}$ (for $1 \le j \le 2^{i+1}$), $A_{i+1,2j-1}, A_{i+1,2j} \subset A_{ij}$, and $\operatorname{dist}(A_{ij}, A_{ik}) \ge R_i$ for all $1 \le j < k \le 2^{i+1}$. Then

$$S := \bigcap_{i=1}^{\infty} \bigcup_{j=1}^{2^{i+1}} A_{ij}$$

is a Cantor dust in X.

Let $0 < r \le R = \operatorname{diam}(X)/2$ and choose an integer $p \ge 0$ so that $R_{p+1} \le r \le R_p = (\vartheta/4)^p R$. Let $\{B_n\}$ be any cover of $\bar{B}(x_0; r)$ by open balls. By compactness we may assume there are only finitely many balls, say $\{B_1, \dots, B_N\}$. Consider a fixed $n \in \{1, \dots, N\}$. Select $i = i(n)$ with $R_{i+1} \le \operatorname{diam}(B_n) < R_i$. Now

$$\frac{1}{2^i} = \left(\frac{\vartheta}{4}\right)^{\alpha i} = R_{i+1}^{\alpha} R^{-\alpha} \left(\frac{4}{\vartheta}\right)^{\alpha} \le \left(\frac{4}{\vartheta}\right)^{\alpha} R^{-\alpha} \left(\operatorname{diam} B_n\right)^{\alpha}.$$

Also, since $\operatorname{dist}(A_{ij}, A_{ik}) \ge R_i > \operatorname{diam} B_n$, $B_n \cap A_{ij} \ne \emptyset$ for exactly one A_{ij}. Because of this and the 'nesting' of the A_{ij}, we see from the above inequality that for all $k \ge i = i(n)$,

$$\#\{j : A_{kj} \cap B_n \ne \emptyset\} \le 2^{k-i} \le \frac{2^k}{R^{\alpha}} \left(\frac{4}{\vartheta}\right)^{\alpha} \left(\operatorname{diam} B_n\right)^{\alpha}.$$

Now fix $k > \max\{p, i(1), \dots, i(N)\}$. Then the total number of A_{kj} inside $A_{p+1,1} = \bar{B}(x_0; R_{p+1}) \subset \bar{B}(x_0; r)$ is estimated by

$$\frac{2^k}{2^{p+1}} \le \#\{j : A_{kj} \subset \bar{B}(x_0; r)\} \le \sum_{n=1}^{N} \#\{j : A_{kj} \cap B_n \ne \emptyset\} \le \sum_{n=1}^{N} \frac{2^k}{R^{\alpha}} \left(\frac{4}{\vartheta}\right)^{\alpha} \left(\operatorname{diam} B_n\right)^{\alpha}.$$

Hence

$$\sum_{n=1}^{N} \left(\operatorname{diam} B_n\right)^{\alpha} \ge \left(\frac{\vartheta}{4}\right)^{\alpha} \frac{R^{\alpha}}{2^{p+1}}.$$

Finally, since $R^{\alpha}/2^p = [(\vartheta/4)^p R]^{\alpha} = R_p^{\alpha} \ge r^{\alpha}$, we conclude that

$$\mathcal{H}^{\alpha}[B(x_0; r)] \ge \frac{r^{\alpha}}{4}.$$

□

4.B. **Gromov Hyperbolicity.** A geodesic metric space is *Gromov hyperbolic* if its geodesic triangles are Δ-*thin* for some $\Delta > 0$, which means that each point on the edge of any geodesic triangle is within distance Δ of some point on one of the other two edges. That is, if $[x, y] \cup [y, z] \cup [z, x]$ is a geodesic triangle, then for all $u \in [x, y]$, $\operatorname{dist}(u, [x, z] \cup [y, z]) \le \Delta$. (Here $[x, y]$ denotes some arbitrary, but fixed, geodesic joining x, y.) Good sources for information concerning Gromov hyperbolicity include [BHK01], [BBI01], [BS00], [BH99], [Bon96] and especially the references mentioned in these works. Roughly speaking, all simply connected manifolds with negative curvature are Gromov hyperbolic; e.g., every CAT(κ) space with $\kappa < 0$. For a specific example, consider any bounded strictly pseudoconvex domain Ω (with sufficiently smooth boundary) in complex n-space together with any of the classical hyperbolic distances h; a result of Balogh and Bonk [BB00] asserts that (Ω, h) is a Gromov hyperbolic space (with $\partial_G \Omega = \partial \Omega$, the Euclidean boundary, and canonical conformal gauge determined by the Carnot-Carathéodory distance on $\partial \Omega$).

The *Gromov boundary* of a proper geodesic Gromov hyperbolic metric space is defined as the set of equivalence classes of geodesic rays, with two such rays being equivalent if they have finite Hausdorff distance. An alternative description can be given in terms of (equivalent) sequences which *converge at infinity*; in particular, this allows us to extend the Gromov product, defined by

$$(x|y)_w := (1/2)(|x - w| + |y - w| - |x - y|) \qquad \text{for points } x, y, w \text{ in the space,}$$

to the boundary (cf. [BH99, pp. 431-436]). This in turn yields a canonically defined conformal gauge on the Gromov boundary generated by the quasimetrics

$$q_{w,\varepsilon}(\xi,\eta) = \exp[-\varepsilon(\xi|\eta)_w] \qquad \text{for points } \xi,\eta \text{ in the boundary.}$$

For $0 < \varepsilon < \varepsilon(\Delta) = \log 2/(4\Delta)$, each quasimetric q_ε is bilipschitz equivalent to an honest metric on the boundary, and all these metric spaces are quasisymmetrically equivalent to each other; in particular they all generate the same topology on the boundary, and the boundary is compact. See Fact 4.1, [BH99, Prop.3.21, pp.435-436], [BHK01, p.18].

4.C. **Modulus.** For $p \geq 1$ we define the *p-modulus* of a family Γ of curves in (X,d,μ) by

$$\operatorname{mod}_p \Gamma = \inf \int \rho^p \, d\mu,$$

where the infimum is taken over all Borel functions $\rho : X \to [0,\infty]$ satisfying $\int_\gamma \rho \, ds \geq 1$ for all locally rectifiable curves $\gamma \in \Gamma$. Then the *p-modulus* of a pair of disjoint compact sets $E,F \subset X$ is

$$\operatorname{mod}_p(E,F;X) = \operatorname{mod}_p \Delta(E,F;X)$$

where $\Delta(E,F;X)$ is the family of all curves joining the sets E,F in X.

For the reader's convenience, we cite the following modulus estimates; see [HK98, 3.14,3.15]. First we have the standard estimate for 'long' curves.

4.3. **Fact.** *Let $x \in X$ and suppose that the upper mass condition $\mu[B(x;R)] \leq MR^p$ holds for some $R > 0$. Let Γ be a family of curves in $B(x;R)$ and suppose that each $\gamma \in \Gamma$ has arclength $\ell(\gamma) \geq L > 0$. Then*

$$\operatorname{mod}_p \Gamma \leq L^{-p}\mu[B(x;R)] \leq M(R/L)^p.$$

Next we record the spherical ring estimate.

4.4. **Fact.** *Let $x \in X$, $0 < 2r \leq R$, and suppose that the upper mass condition $\mu[B(x;t)] \leq Mt^p$ holds for all $0 < t < r + R$. Then*

$$\operatorname{mod}_p(\bar{B}(x;r), \Omega \setminus B(x;R); \Omega) \leq C \left(\log \frac{R}{r}\right)^{1-p},$$

where $C = 2^{p+1}M/\log 2$.

4.D. **Ahlfors Regular and Loewner Spaces.** A metric measure space (X,d,μ) is *Ahlfors Q-regular* provided there exists a finite constant $M = M_\mu$ such that for all $x \in X$ and all $0 < r \leq \operatorname{diam}\Omega$,

$$M^{-1}r^Q \leq \mu[B(x;r)] \leq M\, r^Q.$$

The positive real number Q will then be the Hausdorff dimension of our space, and the Q-dimensional Hausdorff measure $\mathcal{H}^Q$ on X will also satisfy the above inequalities (possibly with a change in the constant M). A metric space (X,d) is Ahlfors Q-regular if $(X,d,\mathcal{H}^Q)$ is Ahlfors Q-regular. We use the adjectives *upper* or *lower* to indicate that only one of these inequalities is in force, and—in the abstract domain setting—add the adjective *locally* to mean that the required inequality holds (or, inequalities hold) for Whitney balls (i.e., for radii $0 < r \leq \lambda d(x)$).

There is an interesting result which gives upper estimates for the Assouad dimension of subsets of Ahlfors regular spaces. See [BHR01, 3.12], [DS97, 5.8], [Luu98, 5.2].

4.5. **Fact.** *Suppose X is an Ahlfors Q-regular space and let $M \subset X$. Then* $\dim_{\mathcal{A}} M < Q$ *if and only if M is porous in X; the constants depend only on each other and the $\mathcal{H}^Q$-regularity constant.*

The notion of a Loewner space was introduced by Heinonen and Koskela in their study [HK98] of quasiconformal mappings of metric spaces; Heinonen's recent monograph [Hei01] renders an enlightening account of these ideas. A path-connected metric measure space (X, d, μ) is a *Q-Loewner space*, $Q > 1$, provided the *Loewner control function*

$$\varphi(t) := \inf\{\mathrm{mod}_Q(E, F; X) : \Sigma(E, F) \leq t\}$$

is strictly positive for all $t > 0$; here E, F are non-degenerate disjoint continua in Ω and

$$\Sigma(E, F) = \mathrm{dist}(E, F)/\min\{\mathrm{diam}(E), \mathrm{diam}(F)\}$$

is the relative size of the pair E, F. Note that we always have $Q \geq \dim_{\mathcal{H}}(\Omega) \geq 1$.

Following [BHK01, p.48], we use the terminology *locally Q-Loewner*, for an abstract domain (Ω, d, μ), to mean that there exists $\kappa \geq 1$, $\varepsilon_0 \in (0, 1/\kappa]$ and a decreasing function $\psi : (0, \infty) \to (0, \infty)$ such that for all $0 < \varepsilon \leq \varepsilon_0$, all $x \in \Omega$ and all non-degenerate disjoint continua E, F in $\varepsilon B(x)$,

$$\Sigma(E, F) \leq t \implies \quad \mathrm{mod}\,_Q(E, F; \kappa\varepsilon B(x)) \geq \psi(t).$$

Bonk, Heinonen and Koskela established the non-trivial result that in a uniform space, this local Loewner condition actually implies a global one [BHK01, Thm.6.4].

4.E. **Doubling Spaces.** A metric space (X, d) satisfies a (metric) *doubling condition* if there is a constant N such that each ball in X of radius R can be covered by at most N balls of radius $R/2$; these are precisely the spaces of finite Assouad dimension. A Borel measure ν is a *doubling measure* on X if there is a constant $D = D_\nu$ such that

$$\nu[B(x; 2r)] \leq D\nu[B(x; r)] \quad \text{for all } x \in X \text{ and all } r > 0.$$

A Borel measure ν on X is *p-homogeneous* if there is a constant $C = C_\nu$ such that

$$\frac{\nu[B(x; R)]}{\nu[B(x; r)]} \leq C \left(\frac{R}{r}\right)^p \quad \text{for all } x \in X \text{ and all } 0 < r \leq R.$$

Obviously every homogeneous measure is doubling; the converse holds too with $C = D$ and $p = \log_2(D)$. Of course Ahlfors regular measures are doubling measures.

The existence of a doubling measure is easily seen to imply a metric doubling condition; the converse holds if our metric space is complete. Here is a precise statement of this result, which is due to Vol'berg and Konyagin for compact spaces, and Luukkainen and Saksman for complete spaces (see [Hei01, Thm.13.5]).

4.6. **Fact.** *A complete doubling space X carries a p-homogeneous measure for each $p >$* $\dim_{\mathcal{A}}(X)$.

An especially important property of doubling measures is their exponential decay on uniformly perfect spaces, which we record as follows; see [Hei01, (13.2)], [Sem99, Lma.B.4.7, p.420].

4.7. **Fact.** *Let ν be a doubling measure on a uniformly perfect metric space. There are constants $C \geq 1$ and $\alpha > 0$, depending only on the doubling constant for ν and the uniformly perfect constant, such that for all balls $B(z;r) \subset B(x;R)$,*

$$\frac{\nu[B(z;r)]}{\nu[B(x;R)]} \leq C\left(\frac{r}{R}\right)^{\alpha}.$$

Now we discuss another way to deform the geometry of a space. Let ν be a doubling measure on a metric space (X,d). For each $\alpha > 0$ we define $\delta = \delta_{\nu,\alpha}$ by

$$\delta(x,y) := \nu[B(xy)]^{\alpha}, \quad \text{where } B(xy) := \bar{B}(x;|x-y|) \cup \bar{B}(y;|x-y|); \tag{4.8}$$

see [DS97, §16.2], [Sem99, (B.3.6)], [Hei01, 14.11]. This always defines a quasimetric on X, and, when X is uniformly perfect, the identity map $(X,d) \to (X,\delta)$ will be quasisymmetric and (X,δ,ν) will be Ahlfors $(1/\alpha)$-regular. Moreover, there is an $\alpha_0 > 0$ (depending only on the doubling constant for ν) such that for all $0 < \alpha < \alpha_0$, $\delta_{\nu,\alpha}$ is bilipschitz equivalent to an honest distance function on X (see Fact 4.1). In particular, if ν is p-homogeneous, then $\delta_{\nu,1/p}$ is already bilipschitz equivalent to an honest distance function (e.g., if (X,d,ν) is Ahlfors p-regular, then $\delta_{\nu,1/p}$ is bilipschitz equivalent to d).

In conjunction with the above chain of ideas, we declare ν to be a *p-dimensional metric doubling measure* on X if ν is a doubling measure on X with the property that $\delta_{\nu,1/p}$ is bilipschitz equivalent to a distance on X. For example, a p-homogeneous measure will be a p-dimensional metric doubling measure. We summarize the above comments; see [Hei01, 14.11,14.14], [Sem99, B.3.7,B.4.6,p.421], [DS97, 16.5,16.7,16.8].

4.9. **Fact.** *Let ν be a p-dimensional metric doubling measure on a uniformly perfect metric space (X,d). Define $\delta = \delta_{\nu,1/p}$ as in* (4.8)*. Then δ is a quasimetric on X which is bilipschitz equivalent to a distance function on X, the identity map $(X,d) \to (X,\delta)$ is quasisymmetric, and (X,δ,ν) is an Ahlfors p-regular space. All of the new parameters depend only on the original data for X and ν.*

There is one final comment we wish to point out regarding the quasimetrics $\delta_{\nu,\alpha}$. As above, suppose ν is a doubling measure on a metric space (X,d), and X has another metric, say, ∂ which is QS equivalent to d. Then by using the doubling property of ν in conjunction with quasisymmetry we see that $\nu[B_d(xy)] \simeq \nu[B_\partial(xy)]$ (where these sets are defined as above using balls centered at x and y in the appropriate metrics); here the constant depends only on the doubling constant and the quasisymmetry data. It therefore follows that the quasimetric δ_d (defined via $B_d(xy)$) is bilipschitz equivalent to δ_∂ (defined via $B_\partial(xy)$).

We note the important fact that quasisymmetric homeomorphisms preserve these doubling conditions; cf. [Hei01, Thm.10.18], [DS97, Lma.16.4]. In particular we see that the notions of doubling measure, the quasimetrics $\delta_{\nu,\alpha}$, and metric doubling measure do not depend on the given distance function per se; they all make sense for a conformal gauge.

4.F. **Quasi Index.** To help the reader unacquainted with our terminology, here we provide the subsection locations for many of our definitions.

2.B	abstract domain	4.D	Ahlfors regular space
4.A	Assouad dimension	3.A	bounded geometry
4.A	conformal Assouad dimension	4.A	conformal gauge
2.E	double cone arc	3.B	doubling metric-density
4.E	doubling space	4.A	geodesic
4.B	Gromov boundary	4.B	Gromov hyperbolic space
4.A	Hausdorff dimension	4.E	homogeneous measure
3.9	lift of metric doubling measure	3.15	lift of metric doubling measure
2.C	linearly locally connected	2.B	locally
4.D	Loewner space	2.C	metric-density
4.E	metric doubling measure	4.C	modulus
2.B	quasiconvex	2.D	quasihyperbolic distance
4.A	quasigeodesic	4.A	quasisymmetry
4.A	roughly starlike	3.9	shadow
3.14	shadow	2.E	uniform arcs and space
2.F	uniformization of BHK	4.A	uniformly perfect
3.A	weakly locally Loewner	2.B	Whitney ball and constant

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