The Standard Matrix of a Rotation

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Linear Transformations are Matrix Transformations

Recall that *every* linear transformation $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ can be written as $\boxed{T(\vec{x}) = A\vec{x}}$ for some $m \times n$ matrix A; A is the *standard matrix* for T.

The j^{th} *column* of A is just $\boxed{\vec{a}_j = T(\vec{e}_j)}$ where

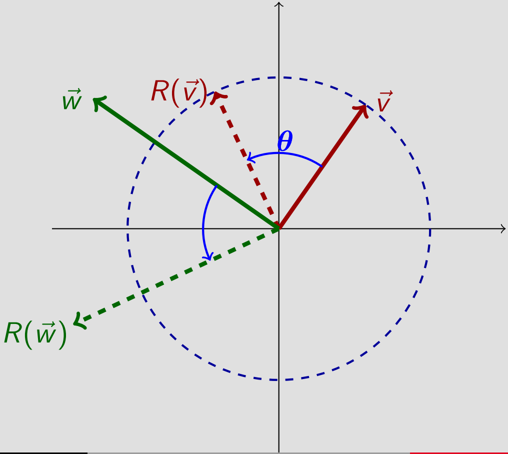
$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

When $m = 2 = n$, so $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^2$, we have

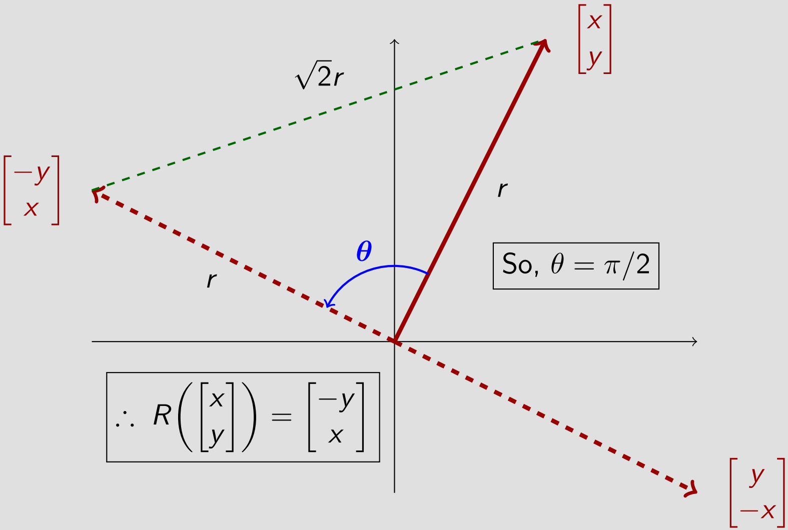
$$A = \begin{bmatrix} T(\vec{e}_1) & T(\vec{e}_2) \end{bmatrix} = \begin{bmatrix} T\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) & T\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \end{bmatrix}.$$

Rotations of the Plane $\mathbb{R}^2$

Let $\mathbb{R}^2 \xrightarrow{R} \mathbb{R}^2$ be the transformation of $\mathbb{R}^2$ given by rotating by θ radians (in the counter-clockwise direction about $\vec{0}$). That is, for each vector $\vec{v}$ in $\mathbb{R}^2$, $R(\vec{v})$ is the result of rotating $\vec{v}$ by θ radians (in the counter-clockwise direction).



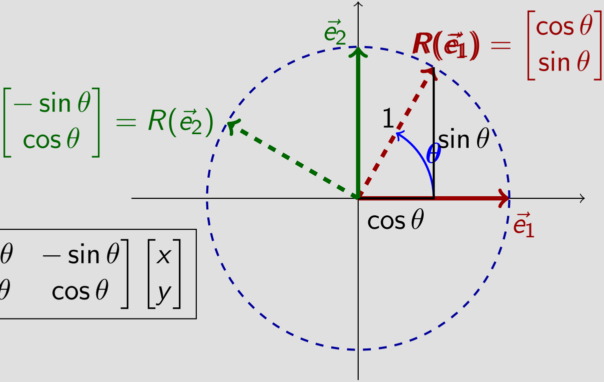
First we examine the special case where we rotate by 90°.



Back to a general rotation $\mathbb{R}^2 \xrightarrow{R} \mathbb{R}^2$ of $\mathbb{R}^2$ by θ radians. We know that

$$R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = A \begin{bmatrix} x \\ y \end{bmatrix} \quad \text{where} \quad A = \begin{bmatrix} R(\vec{e}_1) \ R(\vec{e}_2) \end{bmatrix} = \left[R\left(\begin{bmatrix} 1 \\ 0 \end{bmatrix}\right) \ R\left(\begin{bmatrix} 0 \\ 1 \end{bmatrix}\right) \right].$$

Thus we gotta determine $R(\vec{e}_1)$ and $R(\vec{e}_2)$. This is easy, once we remember a wee bit of trigonometry! ⌣̈



$$\therefore \ R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}$$

Standard Matrix for a Rotation of the Plane $\mathbb{R}^2$

Let $\mathbb{R}^2 \xrightarrow{R} \mathbb{R}^2$ be the transformation of $\mathbb{R}^2$ given by rotating by θ radians (in the counter-clockwise direction about $\vec{0}$).

The *standard matrix* for R is $A = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$.

That is, for all vectors $\begin{bmatrix} x \\ y \end{bmatrix}$ in $\mathbb{R}^2$,

$$\boxed{R\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} x\cos\theta - y\sin\theta \\ x\sin\theta + y\cos\theta \end{bmatrix}}.$$

So, what does the following transformation do?

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix} \mapsto \begin{bmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ 0 & \sin\theta & \cos\theta \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix}$$