Using SVD to "see" a Linear Transformation

Linear Algebra
MATH 2076

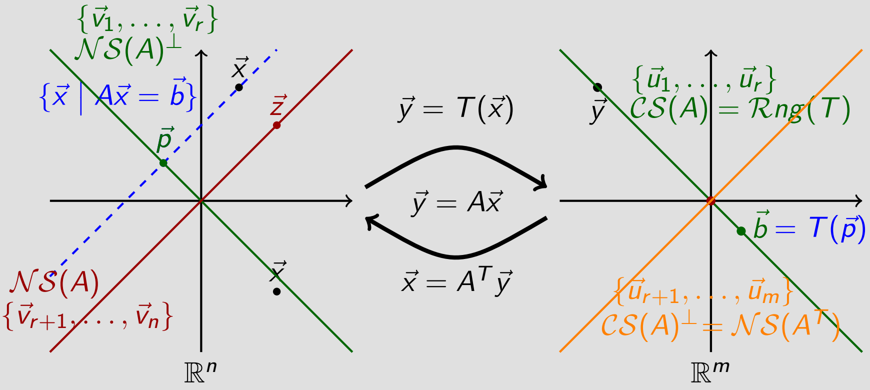
The Four Fundamental Vector SubSpaces Assoc'd with A

The four canonical vector subspaces associated with an $m \times n$ matrix A are:

- the null space $\mathcal{NS}(A)$ of A (a vector subspace of $\mathbb{R}^n$),
- the column space $\mathcal{CS}(A)$ of A (a vector subspace of $\mathbb{R}^m$),
- the orthogonal complement $\mathcal{CS}(A)^\perp = \mathcal{NS}(A^T)$ (a VSS of $\mathbb{R}^m$),
- the orthogonal complement $\mathcal{NS}(A)^\perp = \mathcal{CS}(A^T)$ (a VSS of $\mathbb{R}^n$).

Suppose $A = \left[\vec{u}_1\ \vec{u}_2 \dots \vec{u}_m\right] \Sigma \left[\vec{v}_1\ \vec{v}_2 \dots \vec{v}_n\right]^T$ is an SVD for A. Then:

- $\{\vec{u}_1, \dots, \vec{u}_r\}$ is an *orthonormal basis* for $\mathcal{CS}(A)$,
- $\{\vec{u}_{r+1}, \dots, \vec{u}_m\}$ is an *orthonormal basis* for $\mathcal{CS}(A)^\perp = \mathcal{NS}(A^T)$,
- $\{\vec{v}_{r+1}, \dots, \vec{v}_n\}$ is an *orthonormal basis* for $\mathcal{NS}(A)$, and
- $\{\vec{v}_1, \dots, \vec{v}_r\}$ is an *orthonormal basis* for $\mathcal{NS}(A)^\perp = \mathcal{CS}(A^T)$.



$$A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \ldots & \vec{a}_n \end{bmatrix} \; m \times n \text{ and } \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \text{ given by } T(\vec{x}) = A\vec{x}$$

$\vec{x} = \vec{p} + \vec{z}, \vec{p} = \text{Proj}_{\mathcal{NS}(A)^\perp}(\vec{x}), \vec{z} = \text{Proj}_{\mathcal{NS}(A)}(\vec{x}) \implies A\vec{x} = A\vec{p} + A\vec{z} = A\vec{p}$.
Recall: $\mathcal{NS}(A)^\perp = \mathcal{CS}(A^T)$ and $\mathcal{CS}(A)^\perp = \mathcal{NS}(A^T)$. When have an SVD
$A = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \ldots \vec{u}_m \end{bmatrix} \Sigma \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \ldots \vec{v}_n \end{bmatrix}^T$, get orthon bases $\mathcal{V} = \{\vec{v}_1, \vec{v}_2 \ldots, \vec{v}_n\}$ (for $\mathbb{R}^n$), $\mathcal{U} = \{\vec{u}_1, \vec{u}_2 \ldots, \vec{u}_m\}$ (for $\mathbb{R}^m$), and $\begin{bmatrix} T \end{bmatrix}_{\mathcal{UV}} = \Sigma$. Also, these give. . . .

Example: using an SVD $A = U\Sigma V^T$

Let's try to "see" the linear transformation $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$ where $A = U\Sigma V^T$ is given below.

Here $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$ and we will find U, Σ, V so that $A = U\Sigma V^T$.

Recall that A is the standard matrix for T; i.e., $A = \big[T\big]_{\mathcal{E}}$. It turns out that $\Sigma = \big[T\big]_{\mathcal{U}\mathcal{V}}$ where

$$\mathcal{V} = \{\vec{v}_1, \vec{v}_2\} \quad \text{and} \quad \mathcal{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$$

are the orthonormal bases for $\mathbb{R}^2$ and $\mathbb{R}^3$ (respectively) with

$$V = \big[\vec{v}_1 \;\; \vec{v}_2\big] \quad \text{and} \quad U = \big[\vec{u}_1 \;\; \vec{u}_2 \;\; \vec{u}_3\big].$$

In fact, it even gets better, but lets just work this simple example.

For $A = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$, $A^T A = \begin{bmatrix} 5 & 2 \\ 2 & 8 \end{bmatrix}$ which has evs $\lambda_1 = 9, \lambda_2 = 4$.

Thus $\sigma_1 = 3, \sigma_2 = 2$ and $\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}$. Get assoc'd e$\vec{v}$s $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ so

take $\vec{v}_1 = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} -2 \\ 1 \end{bmatrix}$ and then $V = \frac{1}{\sqrt{5}} \begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$.

Get $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}, A \begin{bmatrix} -2 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ -4 \\ 2 \end{bmatrix}$ so $\vec{u}_1 = \frac{1}{3\sqrt{5}} \begin{bmatrix} 5 \\ 2 \\ 4 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{5}} \begin{bmatrix} 0 \\ -2 \\ 1 \end{bmatrix}$.

Now $\{\vec{u}_3\}$ is a basis for $\mathcal{NS}(A^T)$, so we can take $\vec{u}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ 1 \\ 2 \end{bmatrix}$. Then

$U = \frac{1}{3\sqrt{5}} \begin{bmatrix} 5 & 0 & -2\sqrt{5} \\ 2 & -6 & \sqrt{5} \\ 4 & 3 & 2\sqrt{5} \end{bmatrix}$. $\frac{1}{15} \begin{bmatrix} 5 & 0 & -2\sqrt{5} \\ 2 & -6 & \sqrt{5} \\ 4 & 3 & 2\sqrt{5} \end{bmatrix} \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ -2 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 2 & 0 \\ 0 & 2 \end{bmatrix}$