Using SVD to "see" a Linear Transformation

Linear Algebra
MATH 2076

The Four Fundamental Vector SubSpaces Assoc'd with A

Each $m \times n$ matrix A has four canonical vector subspaces associated with itself. These are:

- the null space $\mathcal{NS}(A)$ of A (a vector subspace of $\mathbb{R}^n$),
- the column space $\mathcal{CS}(A)$ of A (a vector subspace of $\mathbb{R}^m$),
- the orthogonal complement $\mathcal{CS}(A)^\perp = \mathcal{NS}(A^T)$ (a VSS of $\mathbb{R}^m$),
- the orthogonal complement $\mathcal{NS}(A)^\perp = \mathcal{CS}(A^T)$ (a VSS of $\mathbb{R}^n$).

Thus:

- $\{\vec{u}_1, \ldots, \vec{u}_r\}$ is an *orthonormal basis* for $\mathcal{CS}(A)$,
- $\{\vec{u}_{r+1}, \ldots, \vec{u}_m\}$ is an *orthonormal basis* for $\mathcal{CS}(A)^\perp = \mathcal{NS}(A^T)$,
- $\{\vec{v}_{r+1}, \ldots, \vec{v}_n\}$ is an *orthonormal basis* for $\mathcal{NS}(A)$, and
- $\{\vec{v}_1, \ldots, \vec{v}_r\}$ is an *orthonormal basis* for $\mathcal{NS}(A)^\perp = \mathcal{CS}(A^T)$.

Example: using an SVD $A = U\Sigma V^T$

Let's try to "see" the linear transformation $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^3$ defined by $T(\vec{x}) = A\vec{x}$ where $A = U\Sigma V^T$.

Here

$$U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} \quad \text{with} \quad \vec{u}_1 = \frac{1}{\sqrt{2}}\begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{6}}\begin{bmatrix} 1 \\ 2 \\ -1 \end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{3}}\begin{bmatrix} -1 \\ 1 \\ 1 \end{bmatrix}$$

and

$$V = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} \qquad \text{with} \quad \vec{v}_1 = \frac{1}{2}\begin{bmatrix} \sqrt{3} \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{2}\begin{bmatrix} -1 \\ \sqrt{3} \end{bmatrix}$$

and

$$\Sigma = \begin{bmatrix} 3 & 0 \\ 0 & 2 \\ 0 & 0 \end{bmatrix}. \qquad \text{So,} \quad A = \frac{1}{2\sqrt{6}}\begin{bmatrix} 11 & -\sqrt{3} \\ 4 & 4\sqrt{3} \\ 7 & -5\sqrt{3} \end{bmatrix}.$$