The Singular Value Decomposition

Linear Algebra
MATH 2076

University of Cincinnati

Motivation—Why study quadratic functions?

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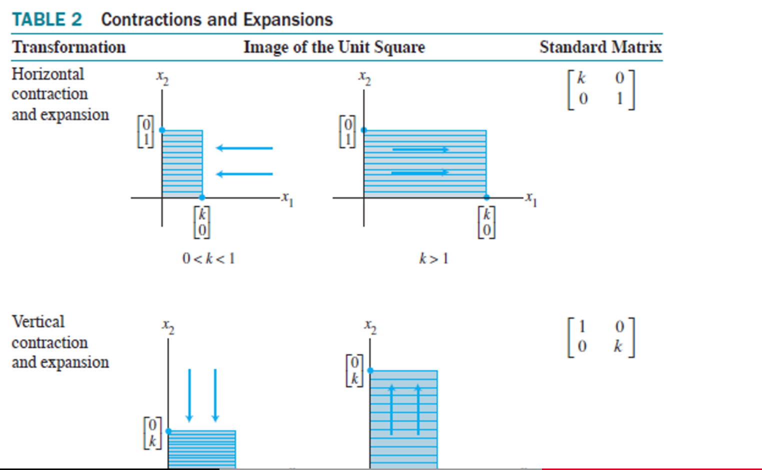

TABLE 2 Contractions and Expansions

Transformation	Image of the Unit Square		Standard Matrix
Horizontal contraction and expansion	$0 < k < 1$	$k > 1$	$\begin{bmatrix} k & 0 \\ 0 & 1 \end{bmatrix}$
Vertical contraction and expansion			$\begin{bmatrix} 1 & 0 \\ 0 & k \end{bmatrix}$

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TABLE 3 Shears

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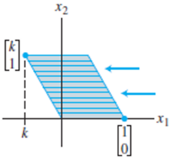
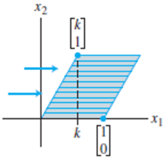
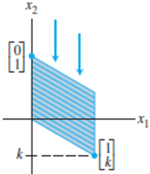
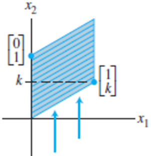
Horizontal shear	 $k < 0$	 $k > 0$	$\begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix}$
Vertical shear			$\begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix}$

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In particular we see that $T(\mathbb{S}^{n-1})$ is *the level set of a quadratic function*!

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Thus, $\mathcal{CS}(A) = \mathcal{S}pan\{A\vec{v}_1, \ldots, A\vec{v}_r\}$. Is $\{A\vec{v}_1, \ldots, A\vec{v}_r\}$ LI?

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$\therefore$ $\{A\vec{v}_1, \ldots, A\vec{v}_r\}$ is an *orthogonal basis* for $\mathcal{CS}(A)$.

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Now define an $r \times r$ diagonal matrix D and an $m \times n$ matrix Σ by

$$\Sigma = \begin{bmatrix} D & \mathbf{0} \\ \mathbf{0} & \mathbf{0} \end{bmatrix} \quad \text{where} \quad D = \begin{bmatrix} \sigma_1 & 0 & \ldots & 0 \\ 0 & \sigma_2 & \ldots & 0 \\ 0 & \vdots & \vdots & 0 \\ 0 & 0 & \ldots & \sigma_r \end{bmatrix}.$$

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Then, $\boxed{A = U\Sigma V^T}$. This is a *singular value decomposition* for A.

Finding matrices U, Σ, V so that $A = U\Sigma V^T$

Start with *any* $m \times n$ matrix A.

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Finding matrices U, Σ, V so that $A = U\Sigma V^T$

Start with *any* $m \times n$ matrix A.

1. Find an orthogonal diagonalization of $M = A^T A$.
2. Write down V and Σ.

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Get $n \times n$ matrix $V = \left[\vec{v}_1 \ \vec{v}_2 \ldots \vec{v}_n\right]$. Also, $\sigma_i = \sqrt{\lambda_i} = \|A\vec{v}_i\|$, which gives D and then Σ.

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Start with *any* $m \times n$ matrix A.

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Need $\{\vec{u}_1, \ldots, \vec{u}_r, \ldots, \vec{u}_m\}$, an orthon basis for $\mathbb{R}^m$. Then have $m \times m$ matrix $U = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \ldots \vec{u}_m \end{bmatrix}$.

For $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$, $A^T A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ which has evs $\lambda_1 = 25, \lambda_2 = 9$.

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For $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$, $A^T A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ which has evs $\lambda_1 = 25, \lambda_2 = 9$.

Thus $\sigma_1 = 5, \sigma_2 = 3$ and $\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. Get assoc'd e$\vec{v}$s $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$

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Get $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}, A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$

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Thus $\sigma_1 = 5, \sigma_2 = 3$ and $\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. Get assoc'd e$\vec{v}$s $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so

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Get $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}, A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ so $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

For $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$, $A^T A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ which has evs $\lambda_1 = 25, \lambda_2 = 9$.

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Need $\vec{u}_3 \cdot \vec{u}_1 = 0 = \vec{u}_3 \cdot \vec{u}_2$.

For $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$, $A^T A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ which has evs $\lambda_1 = 25, \lambda_2 = 9$.

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Get $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$, $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ so $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

Need $\vec{u}_3 \cdot \vec{u}_1 = 0 = \vec{u}_3 \cdot \vec{u}_2$. Use $\vec{u}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$.

For $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$, $A^T A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ which has evs $\lambda_1 = 25, \lambda_2 = 9$.

Thus $\sigma_1 = 5, \sigma_2 = 3$ and $\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. Get assoc'd e$\vec{v}$s $\begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so

take $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and then $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Get $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}$, $A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ so $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}$, $\vec{u}_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

Need $\vec{u}_3 \cdot \vec{u}_1 = 0 = \vec{u}_3 \cdot \vec{u}_2$. Use $\vec{u}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. Then

$$U = \frac{1}{3\sqrt{2}} \begin{bmatrix} 3 & 1 & -2\sqrt{2} \\ 3 & -1 & 2\sqrt{2} \\ 0 & 4 & \sqrt{2} \end{bmatrix}.$$

For $A = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$, $A^T A = \begin{bmatrix} 17 & 8 \\ 8 & 17 \end{bmatrix}$ which has evs $\lambda_1 = 25, \lambda_2 = 9$.

Thus $\sigma_1 = 5, \sigma_2 = 3$ and $\Sigma = \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix}$. Get assoc'd e$\vec{v}$s $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ so

take $\vec{v}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -1 \end{bmatrix}$ and then $V = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}$.

Get $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 5 \\ 5 \\ 0 \end{bmatrix}, A \begin{bmatrix} 1 \\ -1 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$ so $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} 1 \\ -1 \\ 4 \end{bmatrix}$.

Need $\vec{u}_3 \cdot \vec{u}_1 = 0 = \vec{u}_3 \cdot \vec{u}_2$. Use $\vec{u}_3 = \frac{1}{3} \begin{bmatrix} -2 \\ 2 \\ 1 \end{bmatrix}$. Then

$U = \frac{1}{3\sqrt{2}} \begin{bmatrix} 3 & 1 & -2\sqrt{2} \\ 3 & -1 & 2\sqrt{2} \\ 0 & 4 & \sqrt{2} \end{bmatrix}$. $\frac{1}{6} \begin{bmatrix} 3 & 1 & -2\sqrt{2} \\ 3 & -1 & 2\sqrt{2} \\ 0 & 4 & \sqrt{2} \end{bmatrix} \begin{bmatrix} 5 & 0 \\ 0 & 3 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix} = \begin{bmatrix} 3 & 2 \\ 2 & 3 \\ 2 & -2 \end{bmatrix}$