A Quadratic Function Example

Linear Algebra
MATH 2076

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It is simple to use geogebra to "see" the curve $4x^2 + 6xy - 4y^2 = 5$.

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This is easy to check, right?

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To practice your matrix arithmetic skills, check this!

“Finding” the level curve $4x^2 + 6xy - 4y^2 = 5$.

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