Quadratic Functions

Linear Algebra
MATH 2076

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The above objects are are described via *quadratic equations*, which in turn use *quadratic functions*.

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$q(x, y) = x^2 + xy - 4y^2$	with level sets that are what?

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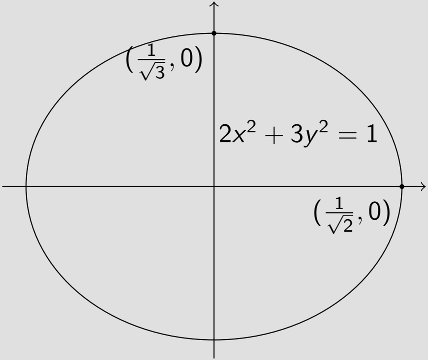

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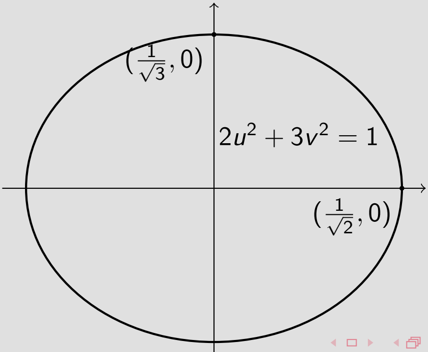

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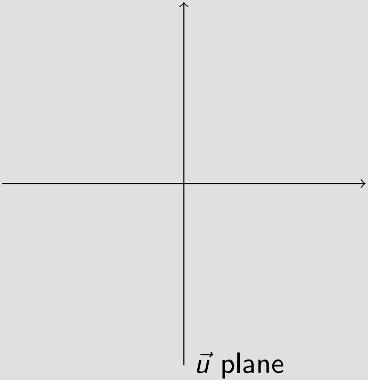

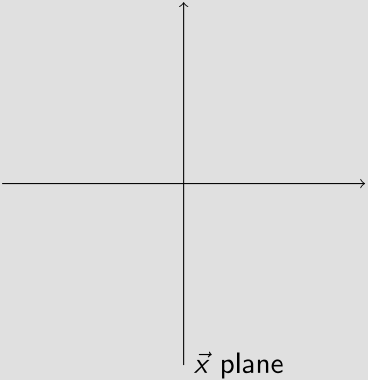

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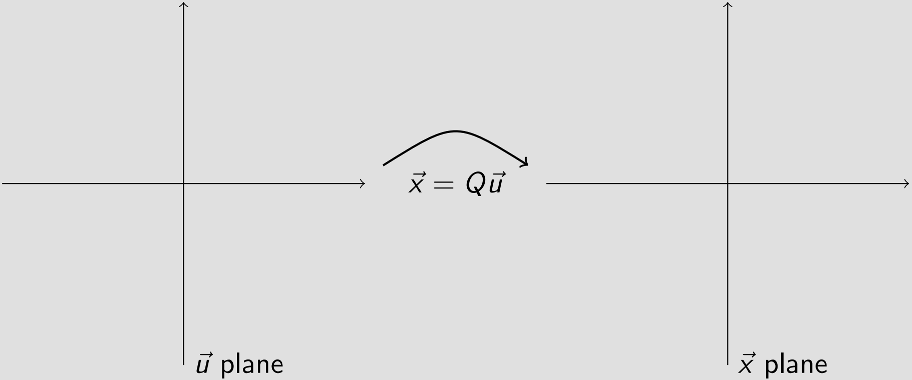

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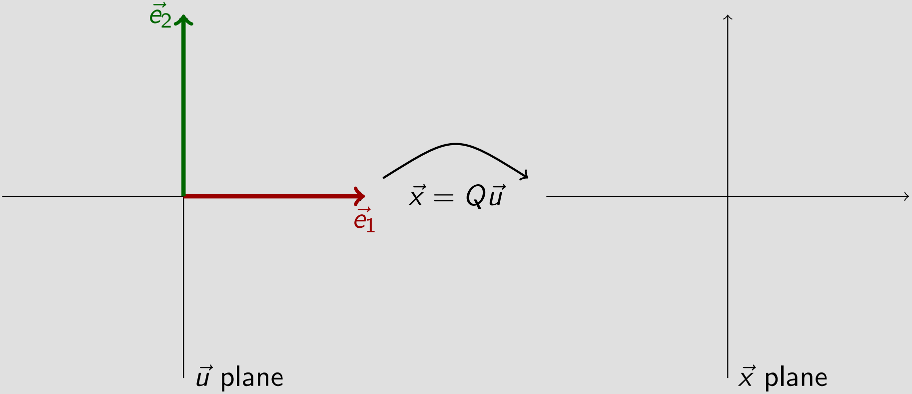

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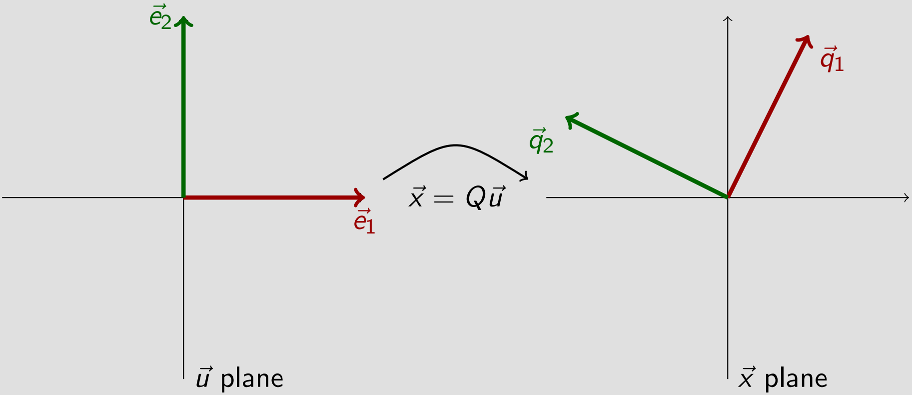

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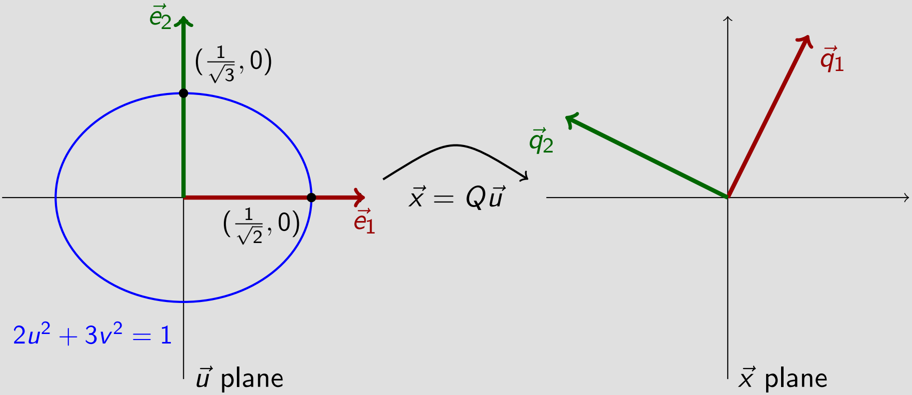

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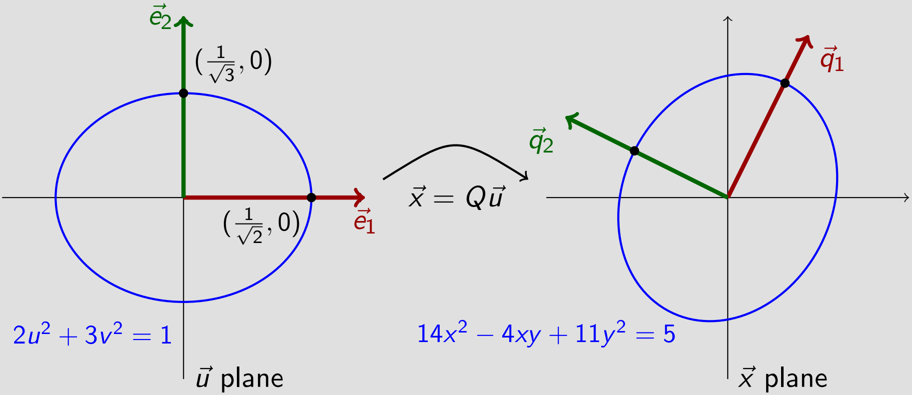

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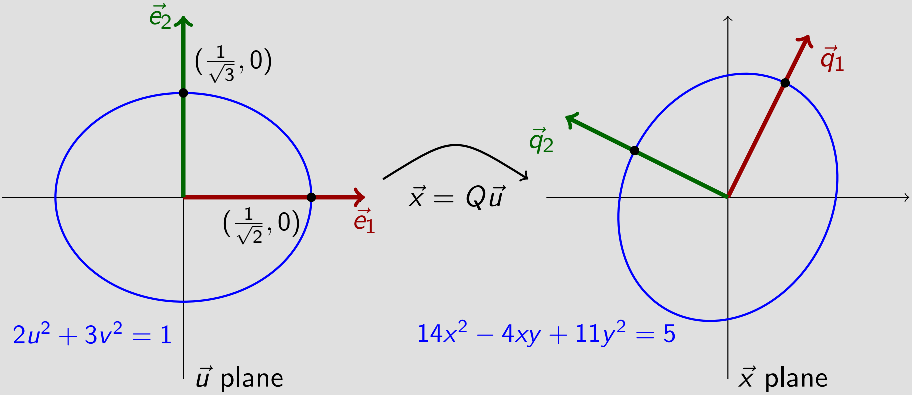

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