Quadratic Functions

Linear Algebra
MATH 2076

From Calculus

In calculus we learn about *linear approximation* (also called *first order approximation*) and study objects such as

- tangent lines,
- tangent planes,
- tangent hyperplanes,
- tangent spaces.

The above objects are all described via *linear equations*.

Sometime we require more information. Then we use a *second order* or *quadratic approximation*. Here we approximate the original graph by objects such as

- subarcs of circles or ellipses or parabolas or hyperbolas,
- quadratic surfaces such as spheres or ellipsoids or paraboloids or hyperboloids.

The above objects are all described via *quadratic equations*, which in turn employ *quadratic functions*.

Some Simple Examples

The function

$$q(x, y) = x^2 + y^2$$

is a quadratic function (of two variables) and the *level set* where $q = 1$ (aka, the *solution set* to the equation $q(x, y) = 1$) is the curve given by the equation

$$x^2 + y^2 = 1,$$

which is the unit circle in the xy-plane. What are the level sets $q = R$ for an arbitrary real number R?

Other examples of quadratic functions include:

$q(x, y) = 4x^2 + 9y^2$ with level sets that are ellipses,

$q(x, y) = x - y^2$ with level sets that are parabolas,

$q(x, y) = x^2 - y^2$ with level sets that are hyperbolas,

$q(x, y) = x^2 + xy - 4y^2$ with level sets that are what?

Quadratic Functions

A *quadratic function* of n-variables is a function $\mathbb{R}^n \xrightarrow{q} \mathbb{R}$ given by

$$\boxed{q(\vec{x}) = \vec{x}^T A \vec{x}} \quad \text{where } A \text{ is an } n \times n \text{ matrix.}$$

Here are some examples with the given 2×2 matrix A:

$$A = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \qquad \text{which gives} \quad q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + 2y^2,$$

$$A = \begin{bmatrix} 4 & 0 \\ 0 & 9 \end{bmatrix} \qquad \text{which gives} \quad q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = 4x^2 + 9y^2,$$

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix} \qquad \text{which gives} \quad q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 - y^2,$$

$$A = \begin{bmatrix} 1 & -2 \\ 3 & -4 \end{bmatrix} \qquad \text{which gives} \quad q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + xy - 4y^2,$$

$$A = \begin{bmatrix} 1 & \frac{1}{2} \\ \frac{1}{2} & -4 \end{bmatrix} \qquad \text{which gives} \quad q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = x^2 + xy - 4y^2.$$

Quadratic Functions

Every quadratic function $\mathbb{R}^n \xrightarrow{q} \mathbb{R}$ given by

$$q(\vec{x}) = \vec{x}^T A \vec{x}$$

can *always* be expressed using a *symmetric* matrix A.

This means that A is orthogonally diagonalizable; we can write $A = QDQ^T$ where Q is orthogonal (so $Q^T Q = I = Q\,Q^T$) and D is diagonal.

The change of variables $\vec{x} = Q\vec{u}$ then gives us

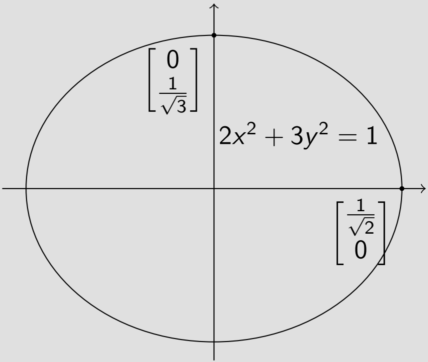
$$\vec{x}^T A \vec{x} = (Q\vec{u})^T A (Q\vec{u}) = \vec{u}^T Q^T A\, Q\vec{u} = \vec{u}^T D\, \vec{u}.$$

It follows that the level set $q(\vec{x}) = 1$ (i.e., $\vec{x}^T A \vec{x} = 1$) can be easily determined once we know the level set $\vec{u}^T D\, \vec{u} = 1$.

A simple 2×2 Example

Let $A = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Then $q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} x & y \end{bmatrix} A \begin{bmatrix} x \\ y \end{bmatrix} = 2x^2 + 3y^2$.

The $q = 1$ level curve is just the solution set for $2x^2 + 3y^2 = 1$. This is an ellipse with vertices $[\frac{\pm 1}{\sqrt{2}} \;\; 0]^T, [0 \;\; \frac{\pm 1}{\sqrt{3}}]^T$ as pictured below.



A 2×2 Example

For $A = \frac{1}{5}\begin{bmatrix} 14 & -2 \\ -2 & 11 \end{bmatrix}$ we get $q\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \frac{1}{5}(14x^2 - 4xy + 11y^2)$. What does $q = 1$ level curve look like? This is given by $14x^2 - 4xy + 11y^2 = 5$.

A has eigenvalues $2, 3$ with associated eigenvectors $\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

These are already orthogonal!

Normalize these eigenvectors to get $\vec{q}_1 = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 \\ 2 \end{bmatrix}, \vec{q}_2 = \frac{1}{\sqrt{5}}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$.

Now we can write

$$A = QDQ^T \quad \text{where} \quad Q = \left[\vec{q}_1 \ \vec{q}_2\right] = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}.$$

Here Q is an *orthogonal* matrix (i.e., $Q^TQ = I$) and therefore $Q^T = Q^{-1}$.

The change of variable $\vec{x} = Q\vec{u}$ helps us to see the picture for the level curve $14x^2 - 4xy + 11y^2 = 5$.

$A = \frac{1}{5}\begin{bmatrix} 14 & -2 \\ -2 & 11 \end{bmatrix} = QDQ^T$ where $Q = \frac{1}{\sqrt{5}}\begin{bmatrix} 1 & -2 \\ 2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$

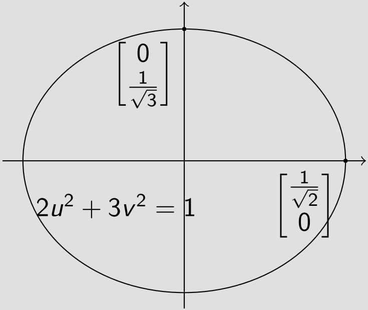
The change of variables $\vec{x} = Q\vec{u}$ then gives us

$$\vec{x}^T A \vec{x} = (Q\vec{u})^T A (Q\vec{u}) = \vec{u}^T Q^T A Q \vec{u} = \vec{u}^T D \vec{u}.$$

To "see" $q(\vec{x}) = 1$ (i.e., $14x^2 - 4xy + 11y^2 = 5$), first draw $\vec{u}^T D \vec{u} = 1$.

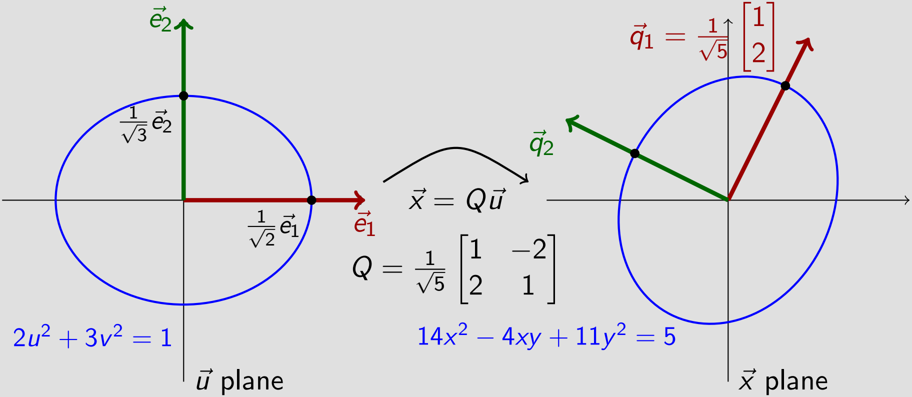
Writing $\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$ we get $\vec{u}^T D \vec{u} = 2u^2 + 3v^2$.

The level curve $\vec{u}^T D \vec{u} = 1$ is given by $2u^2 + 3v^2 = 1$, which is the equation of an ellipse with vertices $[\frac{\pm 1}{\sqrt{2}} \;\; 0]^T, [0 \;\; \frac{\pm 1}{\sqrt{3}}]^T$ as pictured below.



We use pix for $2u^2 + 3v^2 = 1$ to draw pix for $14x^2 - 4xy + 11y^2 = 5$.

We have $\begin{bmatrix} x \\ y \end{bmatrix} = \vec{x} = Q\vec{u} = \begin{bmatrix} u \\ v \end{bmatrix}$ so need both $\vec{u}$-plane and $\vec{x}$-plane.



The standard basis vectors $\vec{e}_1, \vec{e}_2$ are mapped to $\vec{q}_1, \vec{q}_2$. So we see that $\vec{x} = Q\vec{u}$ is a rotation of $\mathbb{R}^2$ by the angle θ where $\tan\theta = 2$. Thus $14x^2 - 4xy + 11y^2 = 5$ is the equation for a *rotated* ellipse with vertices $\dfrac{\pm 1}{\sqrt{2}}\vec{q}_1$ and $\dfrac{\pm 1}{\sqrt{3}}\vec{q}_2$ as pictured above.