Diagonalization of Symmetric Matrices

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

The Spectral Theorem for Symmetric Matrices

Recall that a matrix A is *symmetric* if and only if $A^T = A$.

Theorem

Let A be a symmetric $n \times n$ matrix. Then:

1. *A has n real eigenvalues, counting according to multiplicity.*
2. *For each eigenvalue, the geometric and algebraic multiplicities agree.*
3. *Different eigenspaces are automatically orthogonal.*
4. *A is orthogonally diagonalizable.*

Items (1) and (2) tell us that A is diagonalizable. Thus we can write $A = PDP^{-1}$ where P is invertible and D is diagonal.

Item (3) says that if $\vec{v}$ and $\vec{w}$ are eigenvectors for A associated with different eigenvalues, then $\vec{v} \perp \vec{w}$. This is actually easy to see!

Mutual Orthogonality of Eigenspaces

Suppose $\vec{v}$ and $\vec{w}$ are eigenvectors for a symmetric matrix A associated with different eigenvalues, say $A\vec{v} = \lambda\vec{v}$ and $A\vec{w} = \mu\vec{w}$ with $\lambda \neq \mu$.

Recall that $\vec{v} \cdot \vec{w} = \vec{v}^T\vec{w}$.

Look at

$$\begin{aligned}\lambda\vec{v} \cdot \vec{w} = A\vec{v} \cdot \vec{w} &= \left(A\vec{v}\right)^T\vec{w} = \vec{v}^T A^T\vec{w} \\ &= \vec{v}^T A\vec{w} = \vec{v}^T(\mu\vec{w}) = \mu\vec{v}^T\vec{w} \\ &= \mu\vec{v} \cdot \vec{w}\end{aligned}$$

so

$$(\lambda - \mu)\vec{v} \cdot \vec{w} = 0 \quad \text{and therefore} \quad \vec{v} \cdot \vec{w} = 0.$$

A 3×3 Example

The matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ has *simple* eigenvalues $3, 4, 6$ with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Therefore,

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

Recall that one way to understand this is that $D = [T]_{\mathcal{B}}$ is the $\mathcal{B}$-matrix for the LT $T(\vec{x}) = A\vec{x}$, where $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ (an eigenbasis associated with A). This is because $T(\vec{v}_j) = \lambda_j \vec{v}_j$ where $\lambda_1 = 3, \lambda_2 = 4, \lambda_3 = 6$.

A 3×3 Example

The matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ has *simple* eigenvalues $3, 4, 6$ with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Notice that $\vec{v}_1 \perp \vec{v}_2 \perp \vec{v}_3 \perp \vec{v}_1$. Normalize these eigenvectors to get an *orthonormal* eigenbasis $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$ assoc'd with A. Now we get

$$A = QDQ^T \quad \text{where} \quad Q = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 & \vec{u}_3 \end{bmatrix} = \frac{1}{\sqrt{6}} \begin{bmatrix} \sqrt{2} & -\sqrt{3} & 1 \\ \sqrt{2} & 0 & -2 \\ \sqrt{2} & \sqrt{3} & 1 \end{bmatrix}.$$

Here Q is an *orthogonal* matrix (i.e., $Q^T Q = I$) and therefore $Q^T = Q^{-1}$ and also $D = [T]_{\mathcal{U}}$.

A 2×2 Example

The matrix $A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$ has *simple* eigenvalues $3, -1$ with assoc'd eigenvectors $\begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix}$; we normalize to $\vec{q}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \vec{q}_2 = \frac{1}{\sqrt{2}} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

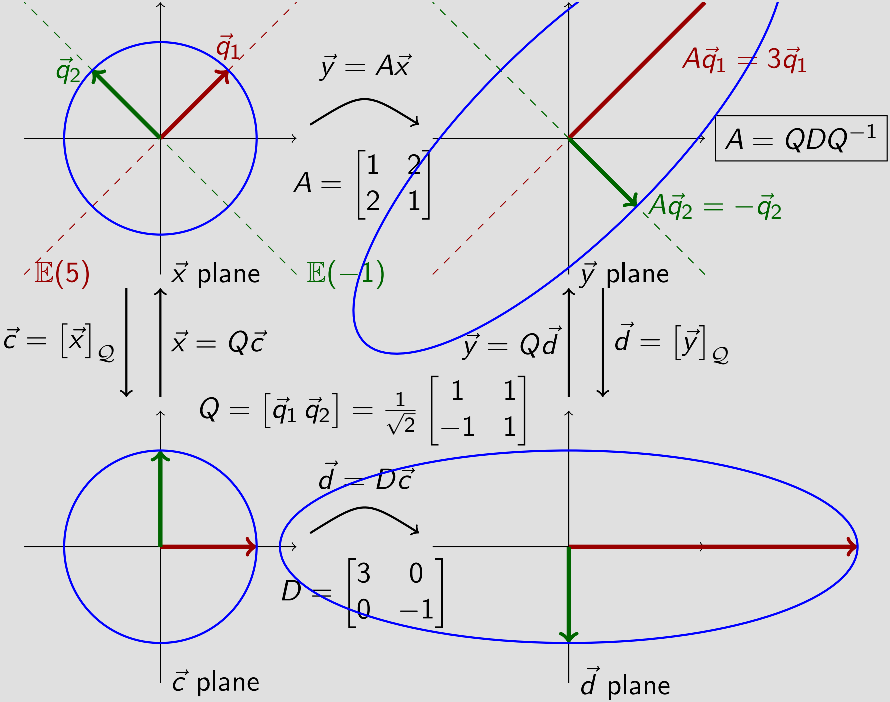
Notice that $\vec{q}_1 \perp \vec{q}_2$ and $\|\vec{q}_1\| = 1 = \|\vec{q}_2\|$, so $\mathcal{Q} = \{\vec{q}_1, \vec{q}_2\}$ is an orthonormal eigenbasis associated with A.

Therefore, we can write

$$A = QDQ^T \quad \text{where} \quad Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 \end{bmatrix} = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}.$$

Here Q is an *orthogonal* matrix (i.e., $Q^T Q = I$) and therefore $Q^T = Q^{-1}$. Notice that multiplication by Q is just ccw rotation by 45°.

The change of variable $\vec{x} = Q\vec{u}$ provides a great picture for the LT $\vec{y} = A\vec{x}$.



$\vec{q}_1$
$\vec{q}_2$
$\vec{y} = A\vec{x}$
$A\vec{q}_1 = 3\vec{q}_1$
$A = QDQ^{-1}$
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
$A\vec{q}_2 = -\vec{q}_2$
$\mathbb{E}(5)$
$\vec{x}$ plane
$\mathbb{E}(-1)$
$\vec{y}$ plane
$\vec{c} = [\vec{x}]_{\mathcal{Q}}$
$\vec{x} = Q\vec{c}$
$\vec{y} = Q\vec{d}$
$\vec{d} = [\vec{y}]_{\mathcal{Q}}$
$Q = [\vec{q}_1\ \vec{q}_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
$\vec{d} = D\vec{c}$
$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$
$\vec{c}$ plane
$\vec{d}$ plane

A 4×4 Example

$A = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$ has simple eigenvalues $9, 5$ and a double eigenvalue 1 with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ & $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

Notice that $\vec{v}_1 \perp \vec{v}_2 \perp \vec{v}_3 \perp \vec{v}_1$ and $\vec{v}_1 \perp \vec{v}_4 \perp \vec{v}_2$ but $\vec{v}_3 \not\perp \vec{v}_4$. Gotta Gram-Schmidt $\{\vec{v}_3, \vec{v}_4\}$. Get new basis $\{\vec{w}_3, \vec{w}_4\}$ where $\vec{w}_3 = \vec{v}_3$ and $\vec{w}_4 = \vec{v}_4 - \mathrm{Proj}_{\vec{v}_3}(\vec{v}_4)$.

Normalize $\{\vec{v}_1, \vec{v}_2, \vec{w}_3, \vec{w}_4\}$ to get an *orthonormal* eigenbasis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4\}$ assoc'd with A. Then we can write

$$A = QDQ^T \quad \text{where} \quad Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \vec{q}_4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A 4×4 Example

For the matrix $A = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$ we can write

$$A = QDQ^T \quad \text{where} \quad Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \vec{q}_4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

which is an *orthogonal* matrix (i.e., $Q^T Q = I$) and therefore $Q^T = Q^{-1}$.

Another 4×4 Example

$A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ has two double eigenvalues $5, 3$ with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ & $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

Notice that $\mathbb{E}(5) \perp \mathbb{E}(3)$ but $\vec{v}_1 \not\perp \vec{v}_2$ and $\vec{v}_3 \not\perp \vec{v}_4$. Gotta Gram-Schmidt both $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_3, \vec{v}_4\}$. Get new *orthogonal* bases $\{\vec{w}_1, \vec{w}_2\}$ for $\mathbb{E}(5)$ and $\{\vec{w}_3, \vec{w}_4\}$ for $\mathbb{E}(3)$.

Normalize $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$ to get an *orthonormal* eigenbasis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4\}$ assoc'd with A. Then we can write

$$A = QDQ^T \quad \text{where} \quad Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \vec{q}_4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Another 4×4 Example

For the matrix $A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ we can write

$$A = QDQ^T \quad \text{where} \quad Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \vec{q}_4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

Here

$$Q = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & -1 & 0 \\ 0 & 1 & 0 & -1 \end{bmatrix}$$

which is an *orthogonal* matrix (i.e., $Q^T Q = I$) and therefore $Q^T = Q^{-1}$.