Diagonalization of Symmetric Matrices

Linear Algebra
MATH 2076

The Spectral Theorem for Symmetric Matrices

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Recall that one way to understand this is that $D = [T]_{\mathcal{B}}$ is the $\mathcal{B}$-matrix for the LT $T(\vec{x}) = A\vec{x}$, where $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ (an eigenbasis associated with A).

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The change of variable $\vec{x} = Q\vec{u}$ provides a great picture for the LT $\vec{y} = A\vec{x}$.

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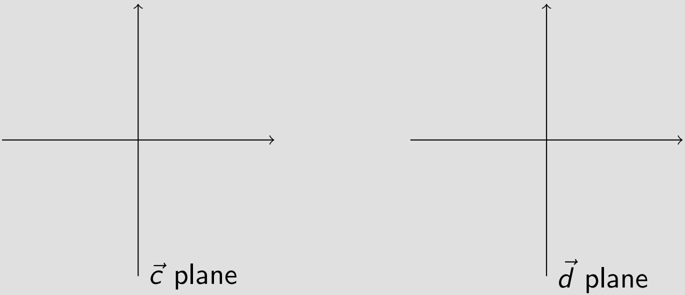

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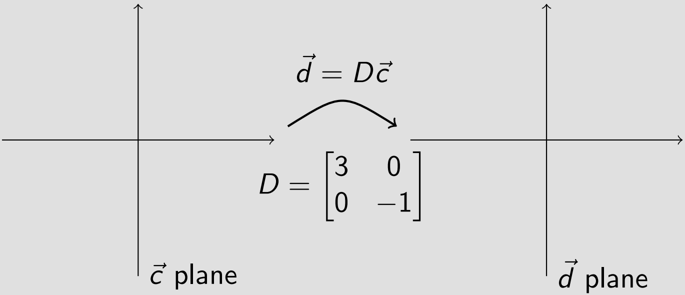
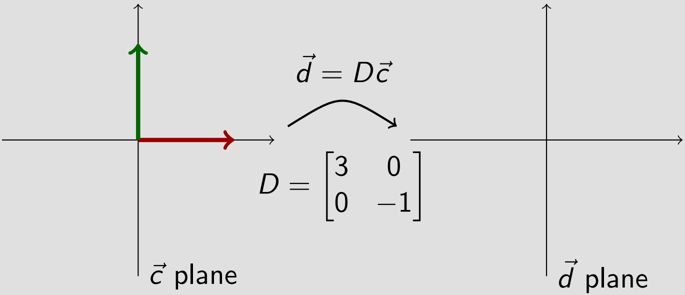

$\vec{d} = D\vec{c}$
$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$
$\vec{c}$ plane
$\vec{d}$ plane

$$\boxed{A = QDQ^{-1}}$$



$\vec{d} = D\vec{c}$
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$\vec{c}$ plane
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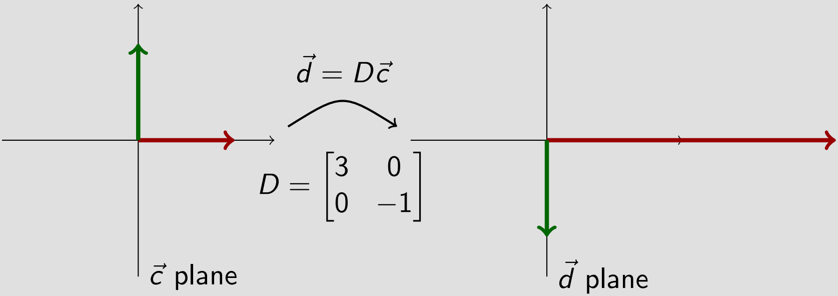
$$A = QDQ^{-1}$$


$\vec{d} = D\vec{c}$
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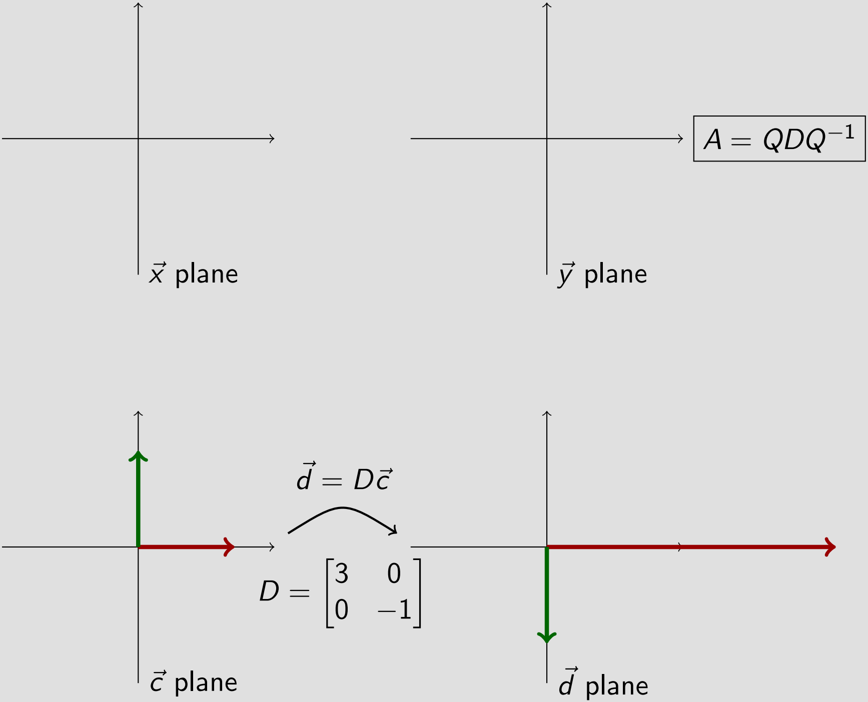
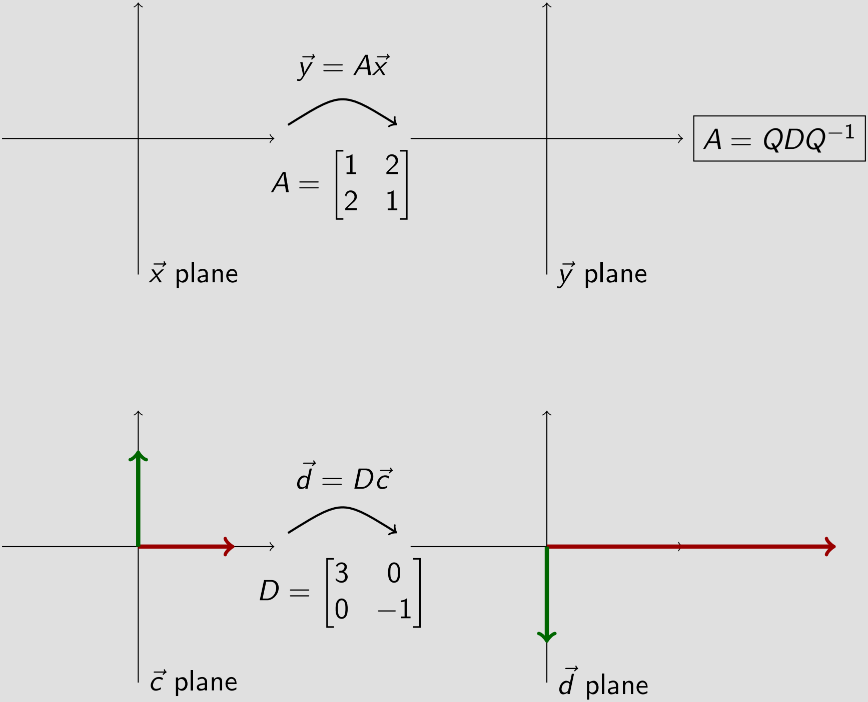
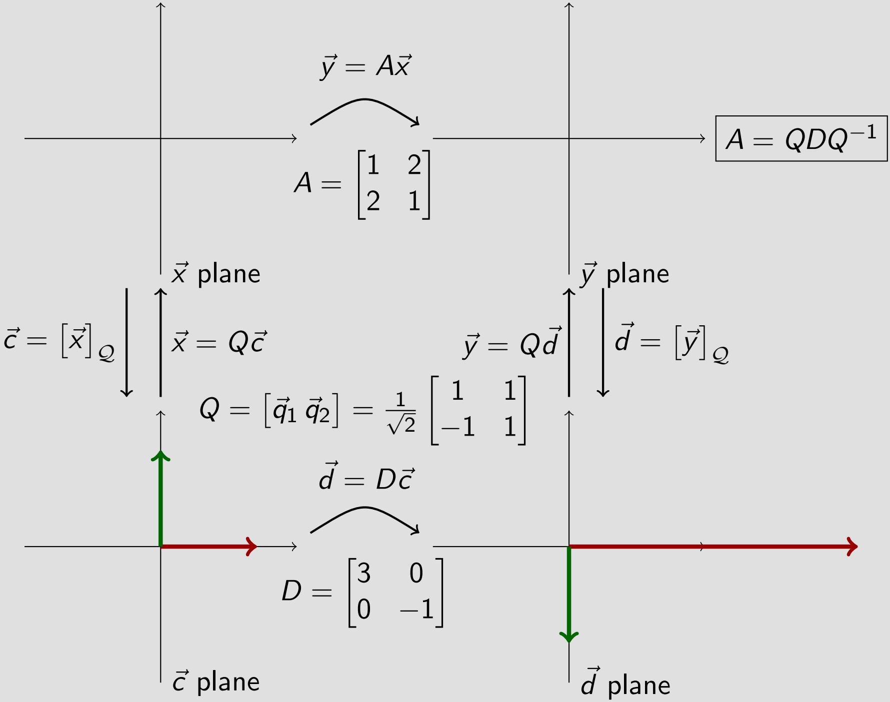
$\vec{c}$ plane
$\vec{d}$ plane



$\vec{x}$ plane
$\vec{y}$ plane
$A = QDQ^{-1}$
$\vec{d} = D\vec{c}$
$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$
$\vec{c}$ plane
$\vec{d}$ plane


$\vec{y} = A\vec{x}$
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
$\vec{x}$ plane
$\vec{y}$ plane
$A = QDQ^{-1}$
$\vec{d} = D\vec{c}$
$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$
$\vec{c}$ plane
$\vec{d}$ plane



$\vec{y} = A\vec{x}$
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
$A = QDQ^{-1}$
$\vec{x}$ plane
$\vec{y}$ plane
$\vec{c} = [\vec{x}]_{\mathcal{Q}}$
$\vec{x} = Q\vec{c}$
$\vec{y} = Q\vec{d}$
$\vec{d} = [\vec{y}]_{\mathcal{Q}}$
$Q = [\vec{q}_1 \; \vec{q}_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
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$D = \begin{bmatrix} 3 & 0 \\ 0 & -1 \end{bmatrix}$
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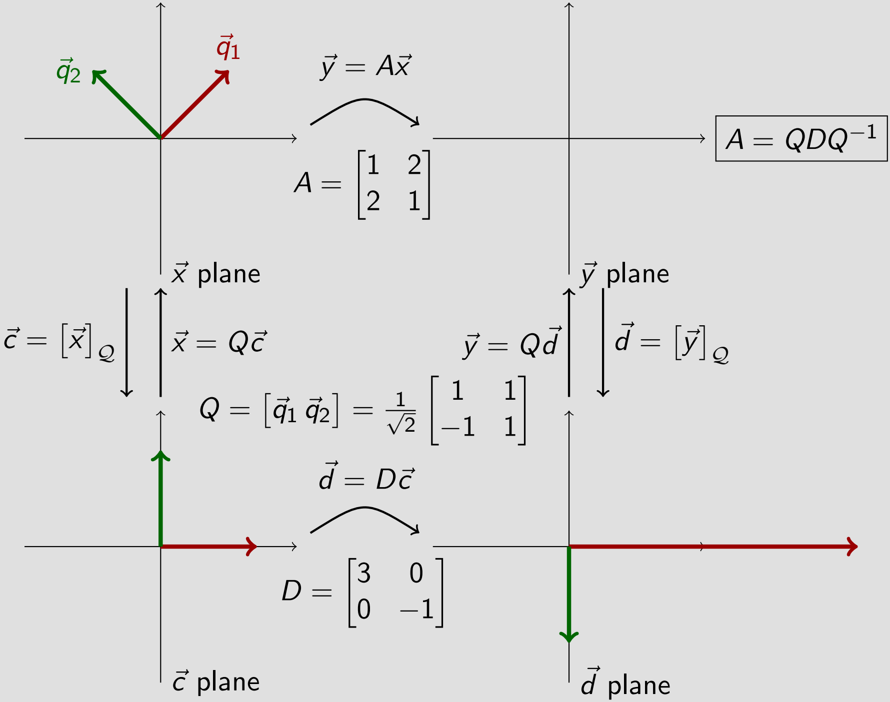

$\vec{q}_1$
$\vec{q}_2$
$\vec{y} = A\vec{x}$
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
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$\vec{x}$ plane
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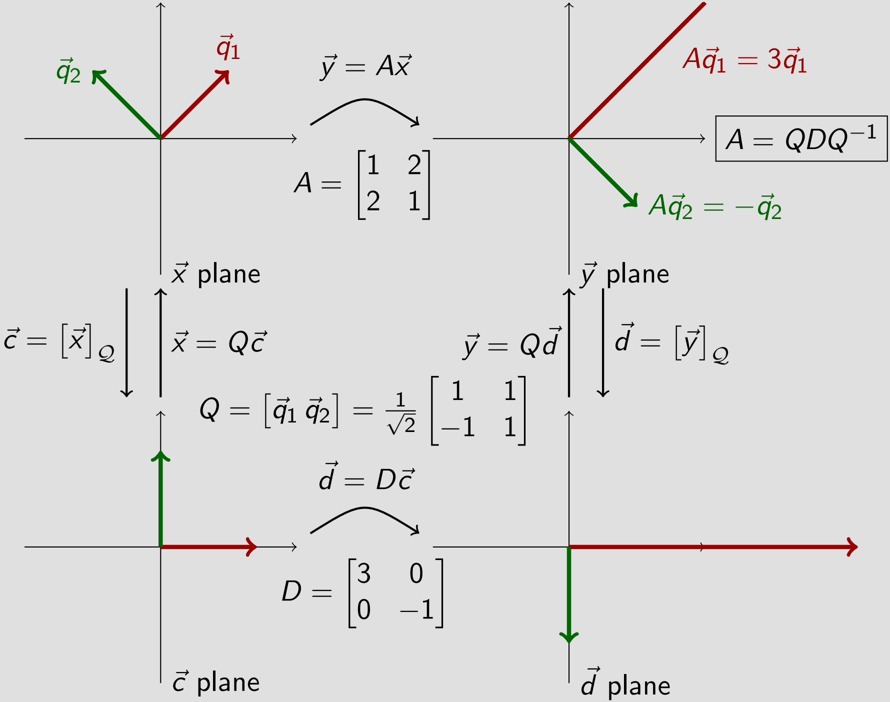
$\vec{q}_2$
$\vec{q}_1$
$\vec{y} = A\vec{x}$
$A\vec{q}_1 = 3\vec{q}_1$
$A = QDQ^{-1}$
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
$A\vec{q}_2 = -\vec{q}_2$
$\vec{x}$ plane
$\vec{y}$ plane
$\vec{c} = [\vec{x}]_{\mathcal{Q}}$
$\vec{x} = Q\vec{c}$
$\vec{y} = Q\vec{d}$
$\vec{d} = [\vec{y}]_{\mathcal{Q}}$
$Q = [\vec{q}_1\ \vec{q}_2] = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$
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$\vec{c}$ plane
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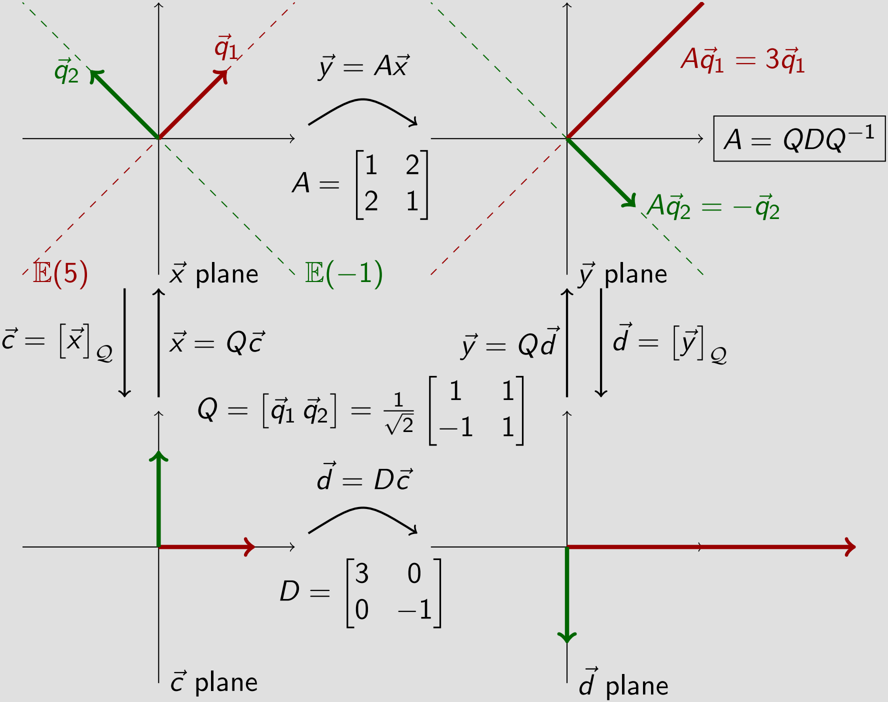
$\vec{q}_2$
$\vec{q}_1$
$\vec{y} = A\vec{x}$
$A\vec{q}_1 = 3\vec{q}_1$
$A = QDQ^{-1}$
$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
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$\mathbb{E}(5)$
$\vec{x}$ plane
$\mathbb{E}(-1)$
$\vec{y}$ plane
$\vec{c} = [\vec{x}]_{\mathcal{Q}}$
$\vec{x} = Q\vec{c}$
$\vec{y} = Q\vec{d}$
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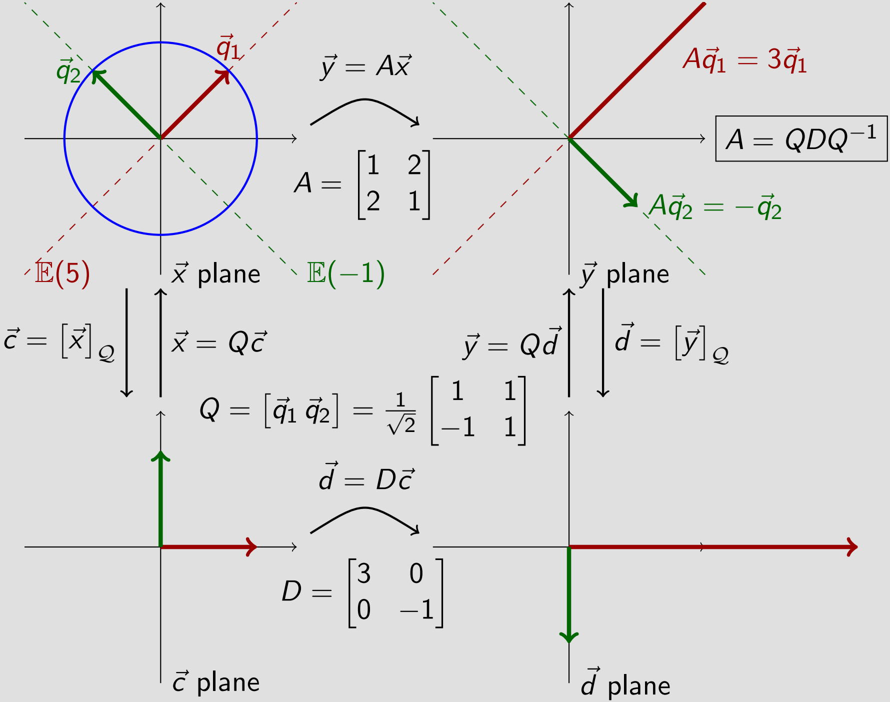

$\vec{q}_1$
$\vec{q}_2$
$\vec{y} = A\vec{x}$
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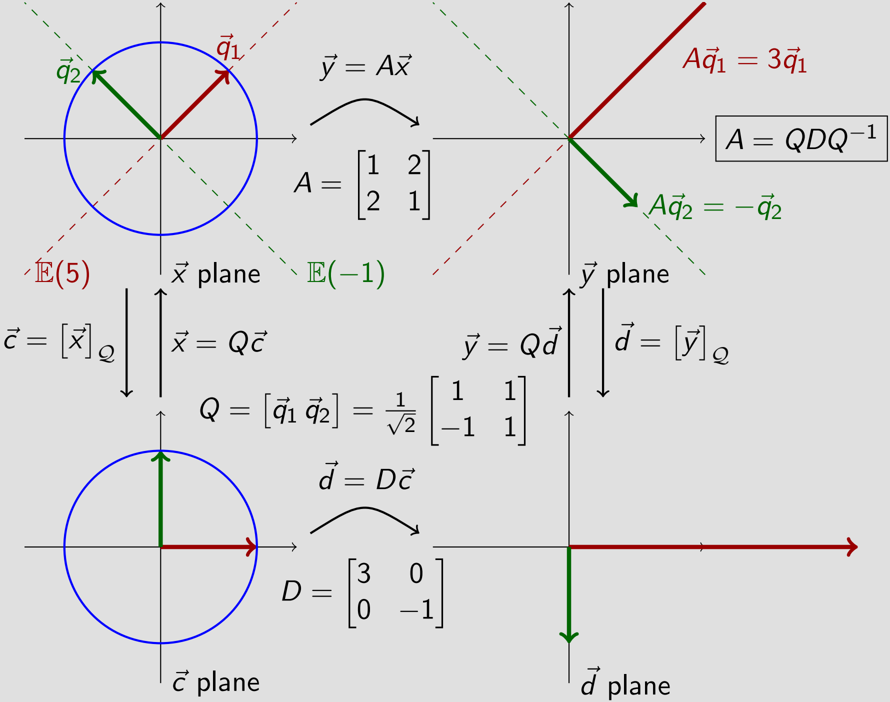
$\vec{q}_1$
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$\vec{y} = A\vec{x}$
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$\vec{x} = Q\vec{c}$
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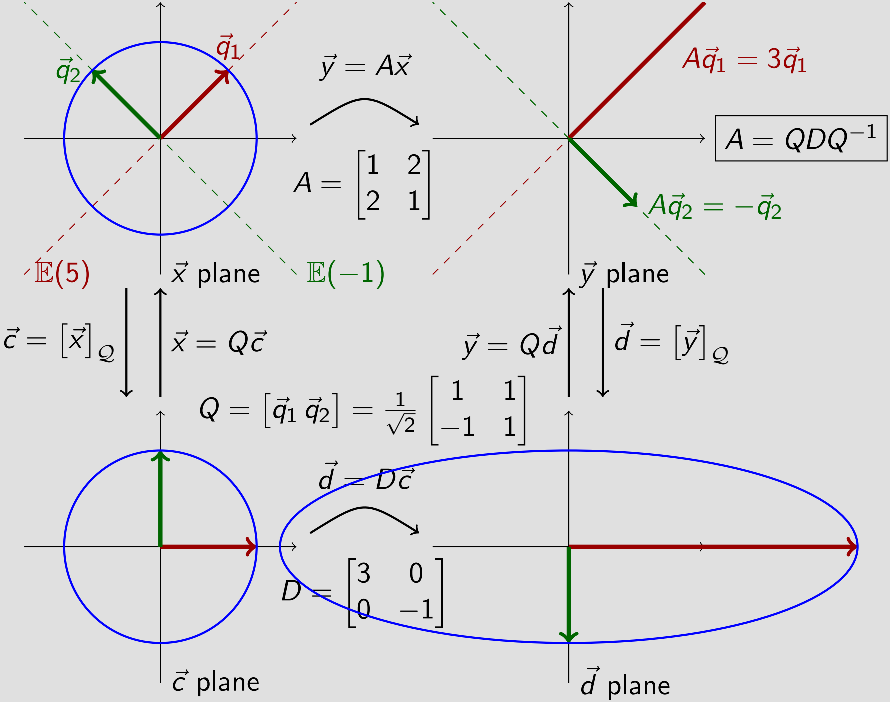
$\vec{c}$ plane
$\vec{d}$ plane



$\vec{q}_2$
$\vec{q}_1$
$\vec{y} = A\vec{x}$
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$A = QDQ^{-1}$
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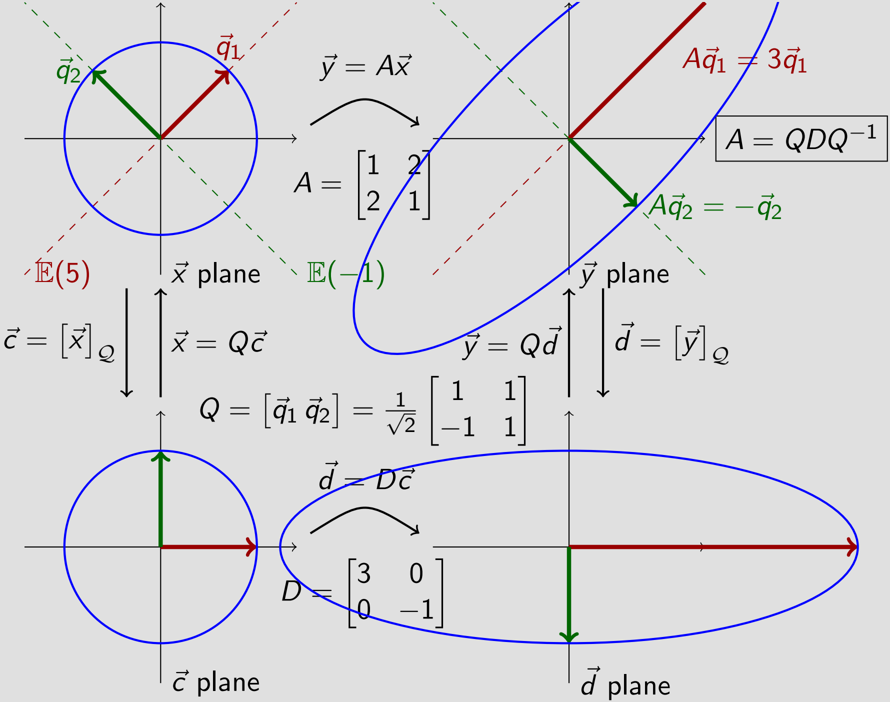

$\vec{q}_1$
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$\vec{y} = A\vec{x}$
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$A = \begin{bmatrix} 1 & 2 \\ 2 & 1 \end{bmatrix}$
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A 4×4 Example

$A = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$ has simple eigenvalues $9, 5$ and a double eigenvalue 1 with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ & $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

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Notice that $\vec{v}_1 \perp \vec{v}_2 \perp \vec{v}_3 \perp \vec{v}_1$ and $\vec{v}_1 \perp \vec{v}_4 \perp \vec{v}_2$ but $\vec{v}_3 \not\perp \vec{v}_4$.

A 4×4 Example

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A 4×4 Example

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Normalize $\{\vec{v}_1, \vec{v}_2, \vec{w}_3, \vec{w}_4\}$ to get an *orthonormal* eigenbasis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4\}$ assoc'd with A.

A 4×4 Example

$A = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$ has simple eigenvalues $9, 5$ and a double eigenvalue 1 with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$ & $\vec{v}_3 = \begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_4 = \begin{bmatrix} 1 \\ -1 \\ 1 \\ -1 \end{bmatrix}$.

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Normalize $\{\vec{v}_1, \vec{v}_2, \vec{w}_3, \vec{w}_4\}$ to get an *orthonormal* eigenbasis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4\}$ assoc'd with A. Then we can write

$$A = QDQ^T \quad \text{where} \quad Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \vec{q}_4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

A 4×4 Example

For the matrix $A = \begin{bmatrix} 4 & 3 & 1 & 1 \\ 3 & 4 & 1 & 1 \\ 1 & 1 & 4 & 3 \\ 1 & 1 & 3 & 4 \end{bmatrix}$

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$$A = QDQ^T \quad \text{where} \quad Q = \begin{bmatrix} \vec{q}_1 \ \vec{q}_2 \ \vec{q}_3 \ \vec{q}_4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 9 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}.$$

Here

$$Q = \frac{1}{2} \begin{bmatrix} 1 & 1 & \sqrt{2} & 0 \\ 1 & 1 & -\sqrt{2} & 0 \\ 1 & -1 & 0 & \sqrt{2} \\ 1 & -1 & 0 & -\sqrt{2} \end{bmatrix}$$

A 4×4 Example

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Here

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which is an *orthogonal* matrix (i.e., $Q^T Q = I$) and therefore $Q^T = Q^{-1}$.

Another 4×4 Example

$A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ has two double eigenvalues $5, 3$ with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ & $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

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Notice that $\mathbb{E}(5) \perp \mathbb{E}(3)$ but $\vec{v}_1 \not\perp \vec{v}_2$ and $\vec{v}_3 \not\perp \vec{v}_4$.

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Another 4×4 Example

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Another 4×4 Example

$A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ has two double eigenvalues $5, 3$ with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ & $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

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Normalize $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$ to get an *orthonormal* eigenbasis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4\}$ assoc'd with A.

Another 4×4 Example

$A = \begin{bmatrix} 4 & 0 & 1 & 0 \\ 0 & 4 & 0 & 1 \\ 1 & 0 & 4 & 0 \\ 0 & 1 & 0 & 4 \end{bmatrix}$ has two double eigenvalues $5, 3$ with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$ & $\vec{v}_3 = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_4 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ -1 \end{bmatrix}$.

Notice that $\mathbb{E}(5) \perp \mathbb{E}(3)$ but $\vec{v}_1 \not\perp \vec{v}_2$ and $\vec{v}_3 \not\perp \vec{v}_4$. Gotta Gram-Schmidt both $\{\vec{v}_1, \vec{v}_2\}$ and $\{\vec{v}_3, \vec{v}_4\}$. Get new *orthogonal* bases $\{\vec{w}_1, \vec{w}_2\}$ for $\mathbb{E}(5)$ and $\{\vec{w}_3, \vec{w}_4\}$ for $\mathbb{E}(3)$.

Normalize $\{\vec{w}_1, \vec{w}_2, \vec{w}_3, \vec{w}_4\}$ to get an *orthonormal* eigenbasis $\{\vec{q}_1, \vec{q}_2, \vec{q}_3, \vec{q}_4\}$ assoc'd with A. Then we can write

$$A = QDQ^T \quad \text{where} \quad Q = \begin{bmatrix} \vec{q}_1 & \vec{q}_2 & \vec{q}_3 & \vec{q}_4 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 & 0 & 0 \\ 0 & 5 & 0 & 0 \\ 0 & 0 & 3 & 0 \\ 0 & 0 & 0 & 3 \end{bmatrix}.$$

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which is an *orthogonal* matrix (i.e., $Q^T Q = I$) and therefore $Q^T = Q^{-1}$.