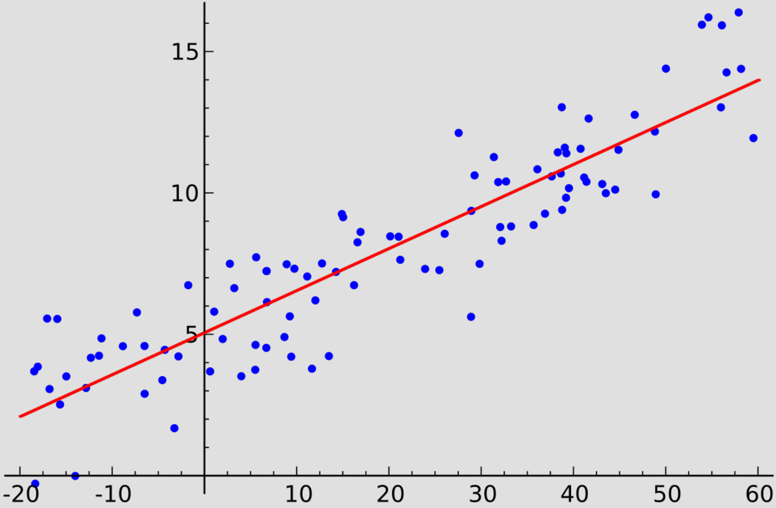
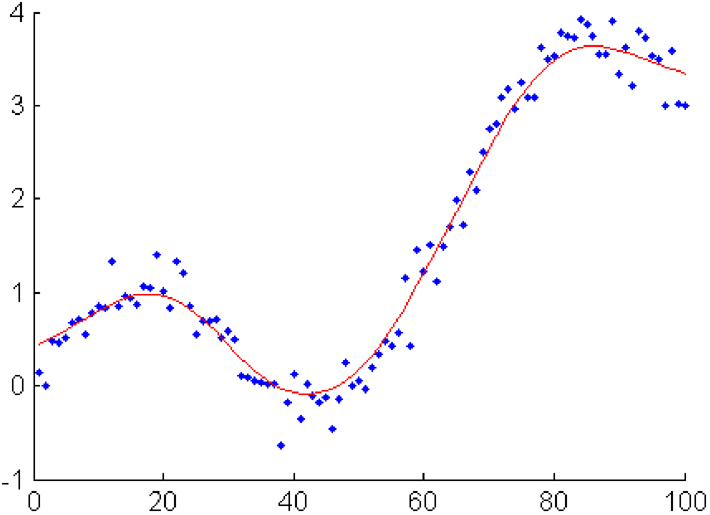
Polynomial Fitting via Least Squares Solutions and the QR Factorization

Linear Algebra
MATH 2076

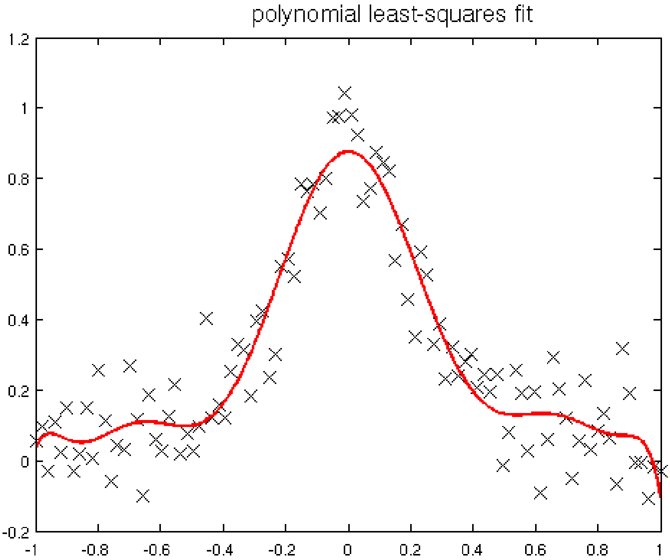
A “Best Fit” Line



A “Best Fit” Polynomial

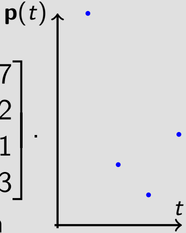


Another "Best Fit" Polynomial



Finding a "Best Fit" Parabola

Let's find the polynomial $\mathbf{p}(t) = c_0 + c_1 t + c_2 t^2$ whose graph "best fits" the data points $(1,7), (2,2), (3,1), (4,3)$.



We seek a least squares solution to $\begin{bmatrix} 1 & 1 & 1 \\ 1 & 2 & 4 \\ 1 & 3 & 9 \\ 1 & 4 & 16 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 7 \\ 2 \\ 1 \\ 3 \end{bmatrix}$.

Modest efforts reveal that the QR-factorization is given via

$$Q = \frac{1}{2} \begin{bmatrix} 1 & -3/\sqrt{5} & 1 \\ 1 & -1/\sqrt{5} & -1 \\ 1 & +1/\sqrt{5} & -1 \\ 1 & +3/\sqrt{5} & 1 \end{bmatrix} \quad \text{and} \quad R = \begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix}.$$

So, we must solve

$$R\vec{c} = Q^T\vec{b} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ \frac{-3}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{-1}{\sqrt{5}} & \frac{3}{\sqrt{5}} \\ 1 & -1 & -1 & 1 \end{bmatrix} \begin{bmatrix} 7 \\ 2 \\ 1 \\ 3 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 13 \\ -13/\sqrt{5} \\ 7 \end{bmatrix}.$$

Finding a "Best Fit" Parabola

We must solve

$$\begin{bmatrix} 2 & 5 & 15 \\ 0 & \sqrt{5} & 5\sqrt{5} \\ 0 & 0 & 2 \end{bmatrix} \begin{bmatrix} c_0 \\ c_1 \\ c_2 \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 13 \\ -13/\sqrt{5} \\ 7 \end{bmatrix}.$$

Clearly $c_2 = 7/4 = 1.75$. Here it is *much easier* to use *back substitution*!
Then $c_1 + 5c_2 = -1.3$, so $c_1 = -10.05$.
Next $c_0 = \frac{13}{4} - \frac{5}{2}c_1 - \frac{15}{2}c_2 = 15.25$.

Thus our "best fit" polynomial is

$$\mathbf{p}(t) = 15.25 - 10.05t + 1.75t^2.$$

Notice that

$$\mathbf{p}(1) = 6.95 \neq 7, \mathbf{p}(4) = 3.05 \neq 3.$$