# The Gram-Schmidt Orthogonalization Procedure

Linear Algebra
MATH 2076

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How do we get $\vec{v}_2, \ldots, \vec{v}_k$?

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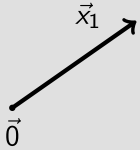

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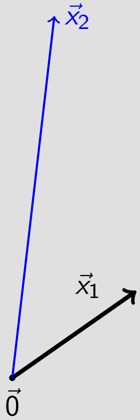

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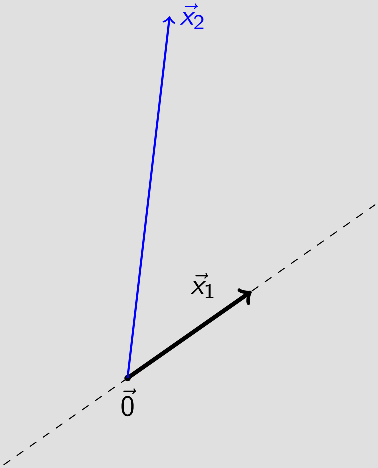

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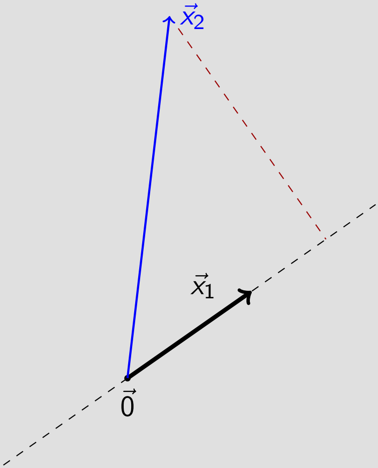

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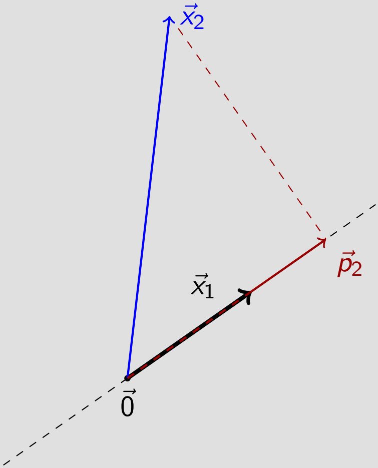

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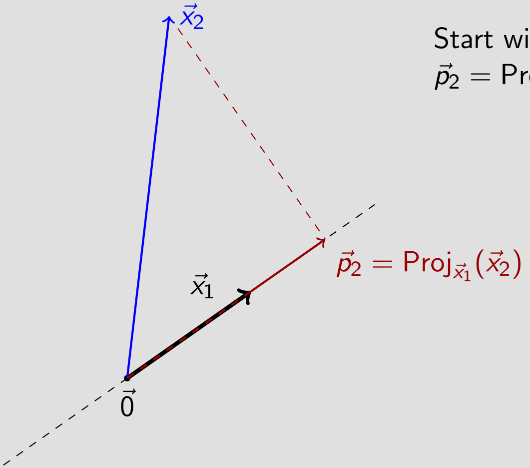


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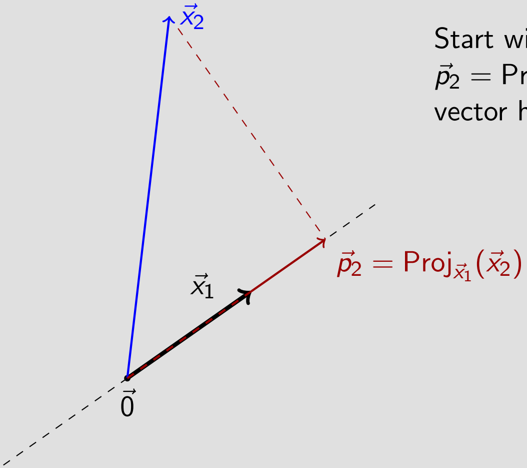


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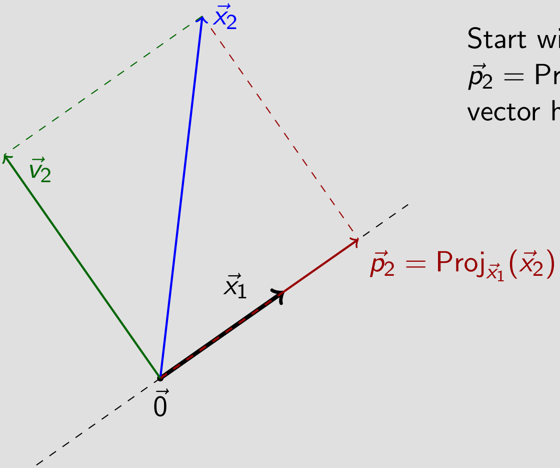


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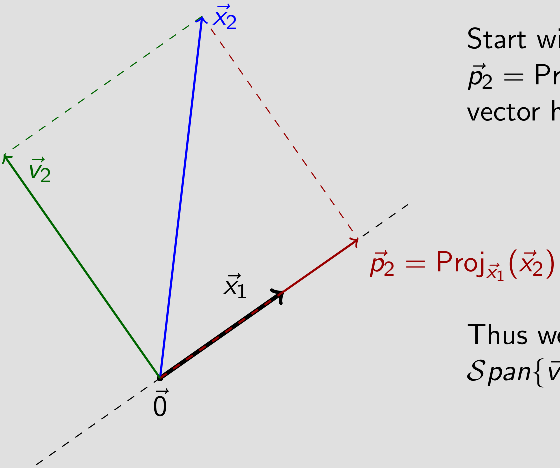


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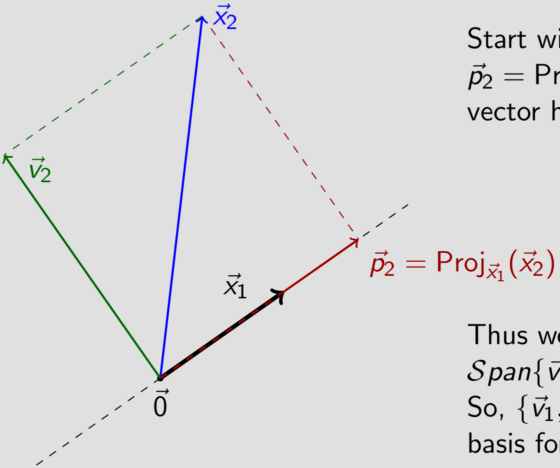


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Let $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ be a basis for some 3-plane $\mathbb{W}$ (in some $\mathbb{R}^n$).

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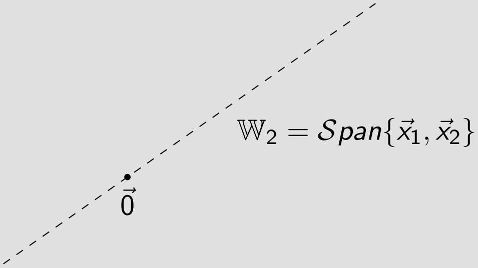

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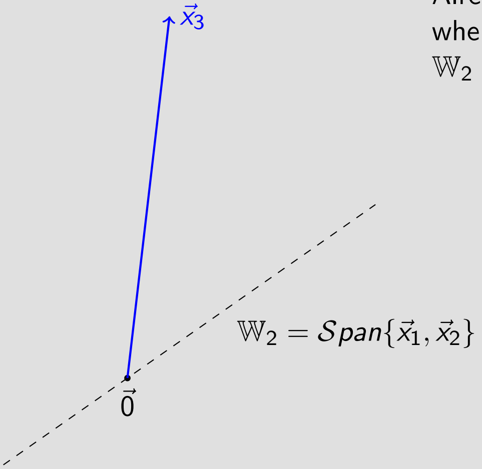


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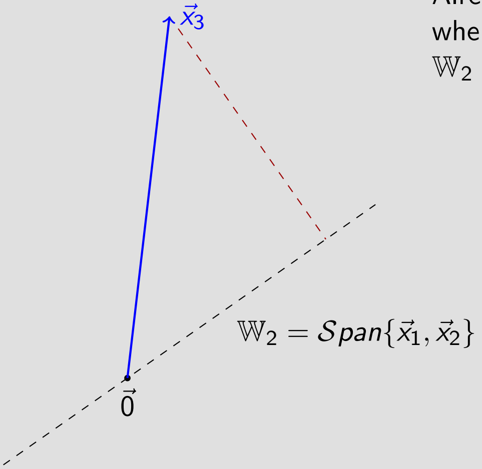


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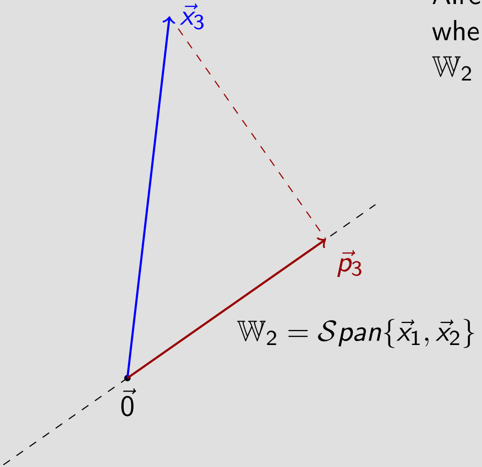


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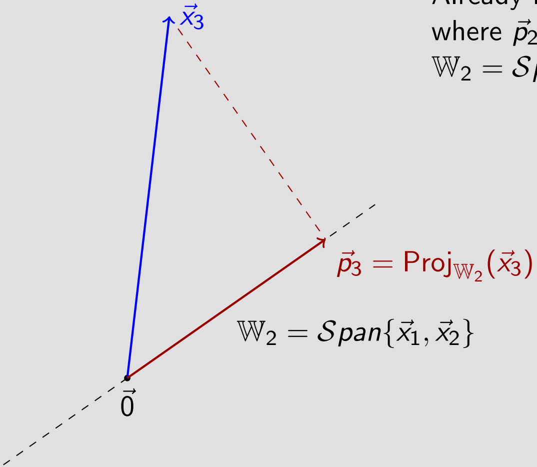

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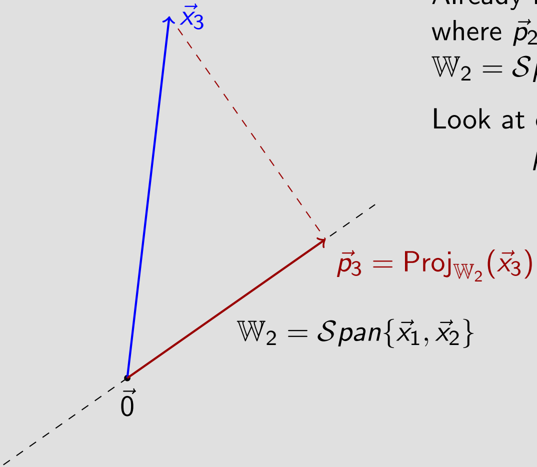


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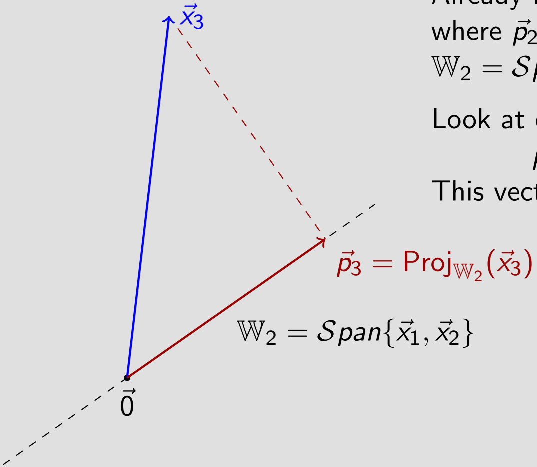


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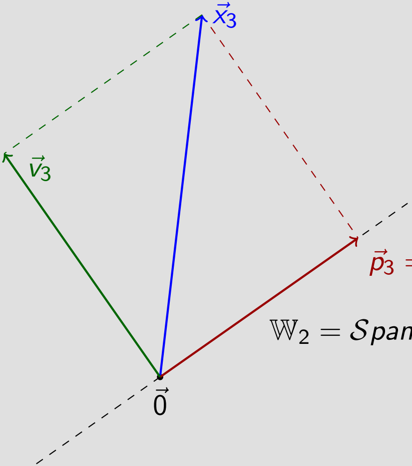


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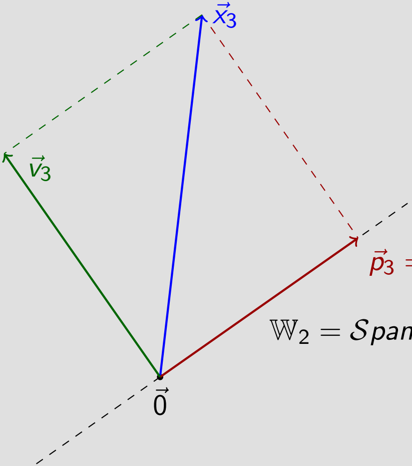


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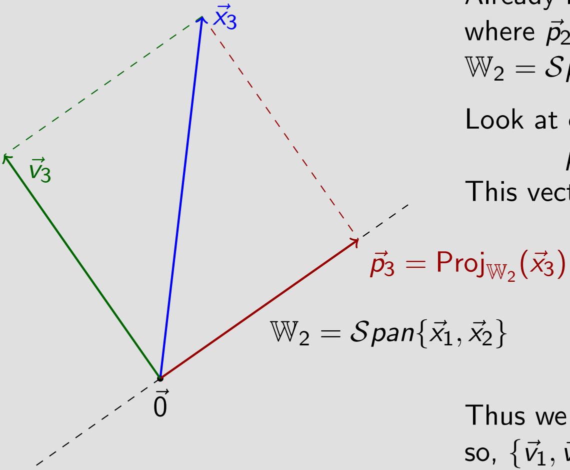


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Start with a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ for some vector space $\mathbb{W}$.
For $1 \leq i < k$, set $\mathbb{W}_i = \mathcal{S}pan\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_i\}$.

Put $\vec{v}_1 = \vec{x}_1$. For $2 \leq i \leq k$:

- Set $\vec{p}_i = \text{Proj}_{\mathbb{W}_{i-1}}(\vec{x}_i) = \sum_{j=1}^{i-1} \text{Proj}_{\vec{v}_j}(\vec{x}_i) = \sum_{j=1}^{i-1} \frac{\vec{x}_i \cdot \vec{v}_j}{\vec{v}_j \cdot \vec{v}_j} \vec{v}_j$.
- Put $\vec{v}_i = \vec{x}_i - \vec{p}_i$.

Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an *orthogonal* basis for $\mathbb{W}$ with

$$\mathbb{W}_i = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i\}.$$

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- Put $\vec{v}_i = \vec{x}_i - \vec{p}_i$.

Then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ is an *orthogonal* basis for $\mathbb{W}$ with

$$\mathbb{W}_i = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_i\}.$$

Finally, we get an *orthonormal* basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ for $\mathbb{W}$ by setting

$$\vec{u}_i = \frac{\vec{v}_i}{\|\vec{v}_i\|} \text{ for } 1 \leq i \leq k.$$

Start with $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$.

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So $\vec{v}_3 = \vec{x}_3 - \vec{x}_1 + \dfrac{2}{3}\vec{v}_2$

and $\vec{v}_3 = \dfrac{1}{3}\begin{bmatrix} 3 \\ -5 \\ 7 \\ -5 \end{bmatrix}$.

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Finally, check orthogonality of

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -3 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 3 \\ -5 \\ 7 \\ -5 \end{bmatrix}.$$

Started with basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ where $\vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 4 \\ 4 \\ 1 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 4 \\ -2 \\ 2 \\ 0 \end{bmatrix}$.

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Got orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where

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Find that $\vec{u}_1 = \frac{1}{2} \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}, \vec{u}_2 = \frac{1}{3\sqrt{2}} \begin{bmatrix} -3 \\ 2 \\ 2 \\ -1 \end{bmatrix}, \vec{u}_3 = \frac{1}{6\sqrt{3}} \begin{bmatrix} 3 \\ -5 \\ 7 \\ -5 \end{bmatrix}$.

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To get orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, just *normalize*!

Find that $\vec{u}_1 = \frac{1}{\sqrt{3}} \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 2 \\ -1 \\ 0 \end{bmatrix}, \vec{u}_3 = \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 2 \end{bmatrix}$.

# The $QR$ Factorization

Let $A$ be a matrix with linearly independent columns, say

$$A = \left[\vec{a}_1\ \vec{a}_2 \dots \vec{a}_k\right] \text{ where } \vec{a}_j = \mathrm{Col}_j(A) \text{ is in } \mathbb{R}^n.$$

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Gram-Schmidt $\mathcal{A}$ to get an orthon basis $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ for $\mathcal{CS}(A)$.

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$$\vec{a}_j = \sum_{i=1}^{k} r_{ij} \vec{u}_i =$$

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$$A = \left[\vec{a}_1\ \vec{a}_2 \ldots \vec{a}_k\right] \text{ where } \vec{a}_j = \text{Col}_j(A) \text{ is in } \mathbb{R}^n.$$

Then $\mathcal{A} = \{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_k\}$ is a basis for the column space $\mathcal{CS}(A)$.

Gram-Schmidt $\mathcal{A}$ to get an orthon basis $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ for $\mathcal{CS}(A)$.
Then each $\vec{a}_j$ can be written as a LC of the $\mathcal{U}$ vectors, so we get

$$\vec{a}_j = \sum_{i=1}^{k} r_{ij}\vec{u}_i = \left[\vec{u}_1\ \vec{u}_2 \ldots \vec{u}_k\right] \begin{bmatrix} r_{1j} \\ r_{2j} \\ \vdots \\ r_{kj} \end{bmatrix}.$$

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Since $\mathcal{U}$ is *orthonormal*, $Q^T Q = I$, and therefore $\boxed{R = Q^T A}$.

# A $QR$ Factorization Example

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