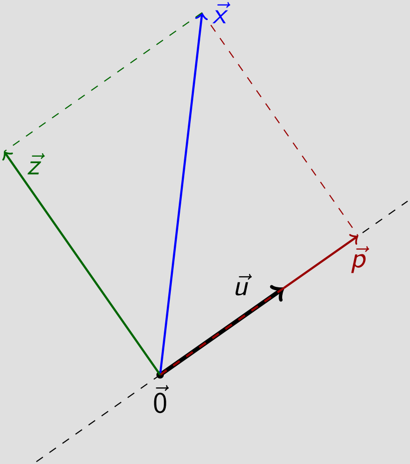
The Gram-Schmidt Orthogonalization Procedure

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Orthogonal Projection Onto a Vector

Let $\vec{u}$ be a fixed vector, and $\vec{x}$ a variable vector.



The *orthogonal projection of $\vec{x}$ onto $\vec{u}$* is the pictured vector $\vec{p}$ which is parallel to $\vec{u}$ (so, $\vec{p} = s\vec{u}$ for some scalar) and has the property that $\vec{z} = \vec{x} - \vec{p} \perp \vec{u}$.
For this to hold, we need $\vec{z} \cdot \vec{u} = 0$, which allows us to determine s. We find that

$$s = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

and thus

$$\vec{p} = \mathrm{Proj}_{\vec{u}}(\vec{x}) = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\,\vec{u}.$$

Note that $\vec{x} = \vec{p} + \vec{z}$ where $\vec{p} \parallel \vec{u}$ and $\vec{z} \perp \vec{u}$.

Orthogonal Projection onto a Vector Subspace $\mathbb{W}$

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \dots, \vec{b}_k\}$ be an orthog basis for a vector subspace $\mathbb{W}$ of $\mathbb{R}^n$.

Theorem (Orthogonal Decomposition Theorem)

Each vector $\vec{x}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\vec{x} = \vec{p} + \vec{z} \quad \textit{where } \vec{p} \textit{ is in } \mathbb{W} \textit{ and } \vec{z} \textit{ is in } \mathbb{W}^{\perp}.$$

In fact,

$$\vec{p} = \sum_{i=1}^{k} \mathrm{Proj}_{\vec{b}_i}(\vec{x}) = \sum_{i=1}^{k} \frac{\vec{x} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i} \vec{b}_i \quad \textit{and } \vec{z} = \vec{x} - \vec{p}.$$

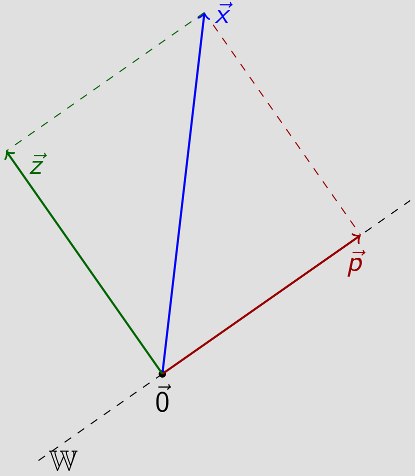
Definition

We call $\vec{p}$ the *orthogonal projection of $\vec{x}$ onto $\mathbb{W}$*, and write $\vec{p} = \mathrm{Proj}_{\mathbb{W}}(\vec{x})$.

Note that we need an *orthogonal basis* $\mathcal{B}$ to compute $\mathrm{Proj}_{\mathbb{W}}(\vec{x})$.

Orthogonal Projection Onto a Vector Subspace

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an orthog basis for a vector subspace $\mathbb{W}$ of $\mathbb{R}^n$, and $\vec{x}$ be any vector in $\mathbb{R}^n$.



The *orthogonal projection of $\vec{x}$ onto $\mathbb{W}$* is the pictured vector $\vec{p}$ which lies in $\mathbb{W}$ and has the property that

$$\vec{z} = \vec{x} - \vec{p} \perp \mathbb{W}.$$

Recall that

$$\vec{p} = \mathrm{Proj}_{\mathbb{W}}(\vec{x}) = \sum_{i=1}^{k} \vec{p}_i$$

where

$$\vec{p}_i = \mathrm{Proj}_{\vec{b}_i}(\vec{x}) = \frac{\vec{x} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i}\,\vec{b}_i.$$

Note that we need an *orthogonal basis* $\mathcal{B}$ to compute $\mathrm{Proj}_{\mathbb{W}}(\vec{x})$.

Examples

Find orthogonal projection onto $\mathbb{W} = \{x_1 + x_2 + x_3 = 0\}$.

Easy to get basis for $\mathbb{W}$, but how to get orthogonal basis?

Simplest way to do this problem is to find orthogonal projection onto $\mathbb{W}^\perp = \mathcal{S}pan\{\vec{n}\}$ where $\vec{n} = [1\ 1\ 1]^T = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$. Then use fact that $\text{Proj}_{\mathbb{W}} + \text{Proj}_{\mathbb{W}^\perp} = \text{Id}$, so $\text{Proj}_{\mathbb{W}}(\vec{x}) = \vec{x} - \text{Proj}_{\mathbb{W}^\perp}(\vec{x})$.

Find orthogonal projection onto

$$\mathbb{W} = \{x_1 + x_2 + x_3 + x_4 = 0, x_2 + x_3 + x_4 = 0\} = \mathcal{NS}\left(\begin{bmatrix} 1 & 1 & 1 & 1 \\ 0 & 1 & 1 & 1 \end{bmatrix}\right).$$

Easy to get basis for $\mathbb{W}$, but how to get orthogonal basis?

Here $\mathbb{W}$ is a 2-plane in $\mathbb{R}^4$, so $\mathbb{W}^\perp$ is also a 2-plane in $\mathbb{R}^4$.

Given a basis, how do we get an orthogonal basis?

First Look at Gram-Schmidt Orthogonalization Procedure

This is an algorithm to produce an orthonormal basis from a basis.

We start with a basis $\{\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_k\}$ for some vector space $\mathbb{W}$.

Then we construct an orthogonal basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k\}$ for $\mathbb{W}$ with certain nice properties.

Finally, we get an orthonormal basis $\{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ for $\mathbb{W}$.

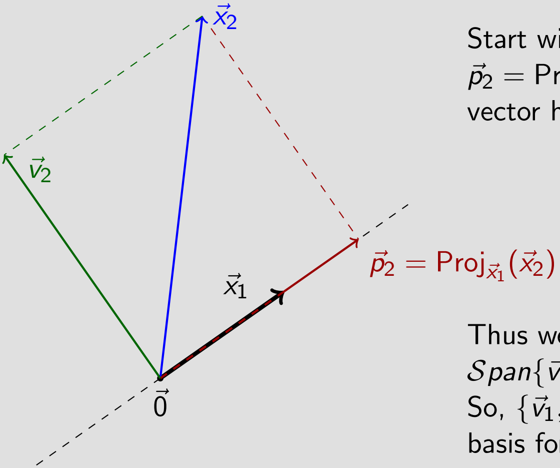
Normalization Step: For $1 \leq i \leq k$, $\vec{u}_i = \dfrac{\vec{v}_i}{\|\vec{v}_i\|}$.

First Step: $\vec{v}_1 = \vec{x}_1$.

How do we get $\vec{v}_2, \ldots, \vec{v}_k$?

Example with Basis $\{\vec{x}_1, \vec{x}_2\}$

Let $\{\vec{x}_1, \vec{x}_2\}$ be a basis for some 2-plane $\mathbb{W}$ (in some $\mathbb{R}^n$).



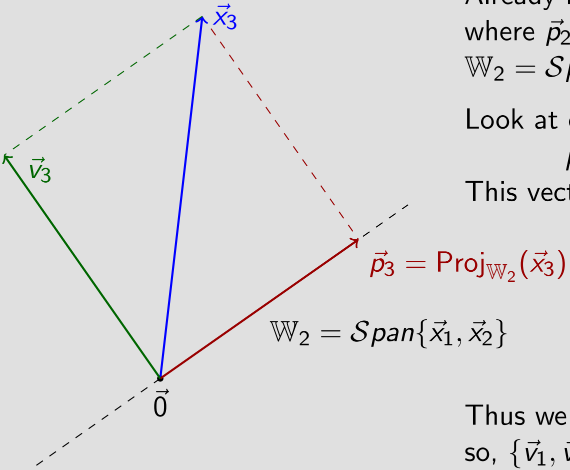
Start with orthogonal projection $\vec{p}_2 = \text{Proj}_{\vec{x}_1}(\vec{x}_2)$ of $\vec{x}_2$ onto $\vec{x}_1$. This vector has the property that

$$\vec{v}_2 = \vec{x}_2 - \vec{p}_2 \perp \vec{x}_1.$$

Thus we find that $\vec{v}_1 = \vec{x}_1 \perp \vec{v}_2$ and $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2\} = \mathcal{S}pan\{\vec{x}_1, \vec{x}_2\} = \mathbb{W}$. So, $\{\vec{v}_1, \vec{v}_2\}$ is the desired orthogonal basis for $\mathbb{W}$.

Example with Basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$

Let $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ be a basis for some 3-plane $\mathbb{W}$ (in some $\mathbb{R}^n$).



Already have $\vec{v}_1 = \vec{x}_1$ and $\vec{v}_2 = \vec{x}_2 - \vec{p}_2$ where $\vec{p}_2 = \text{Proj}_{\vec{x}_1}(\vec{x}_2)$. Also know that $\mathbb{W}_2 = \mathcal{S}pan\{\vec{x}_1, \vec{x}_2\} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2\}$.

Look at orthogonal projection

$$\vec{p}_3 = \text{Proj}_{\mathbb{W}_2}(\vec{x}_3) \text{ of } \vec{x}_3 \text{ onto } \mathbb{W}_2.$$

This vector has the property that

$$\vec{v}_3 = \vec{x}_3 - \vec{p}_3 \perp \mathbb{W}_2.$$

Thus we find that $\vec{v}_1 \perp \vec{v}_2 \perp \vec{v}_3 \perp \vec{v}_1$, so, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the desired orthogonal basis for $\mathbb{W}$.

Start with $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$. First, $\vec{v}_1 = \vec{x}_1$.

Next, $\vec{v}_2 = \vec{x}_2 - \vec{p}_2$ where $\vec{p}_2 = \text{Proj}_{\vec{v}_1}(\vec{x}_2) = \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\,\vec{v}_1 = \dfrac{2}{2}\vec{v}_1 = \vec{x}_1$, so

$\vec{v}_2 = \vec{x}_2 - \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$. Finally, $\vec{v}_3 = \vec{x}_3 - \vec{p}_3$ where $\vec{p}_3 = \text{Proj}_{\mathbb{W}_2}(\vec{x}_3)$.

Here $\vec{p}_3 = \text{Proj}_{\mathbb{W}_2}(\vec{x}_3) = \text{Proj}_{\vec{v}_1}(\vec{x}_3) + \text{Proj}_{\vec{v}_2}(\vec{x}_3)$. Now

$\text{Proj}_{\vec{v}_1}(\vec{x}_3) = \dfrac{\vec{x}_3 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\,\vec{v}_1 = \dfrac{2}{2}\vec{v}_1 = \vec{x}_1$ and $\text{Proj}_{\vec{v}_2}(\vec{x}_3) = \dfrac{\vec{x}_3 \cdot \vec{v}_2}{\vec{v}_2 \cdot \vec{v}_2}\,\vec{v}_2 = 0\vec{v}_2 = \vec{0}$.

Thus $\vec{p}_3 = \vec{x}_1$, so

$$\vec{v}_3 = \vec{x}_3 - \vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$$

Finally, check that

$$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$$

are orthogonal.

Started with basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ where $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Got orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

To get orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, just *normalize*!

Get $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.