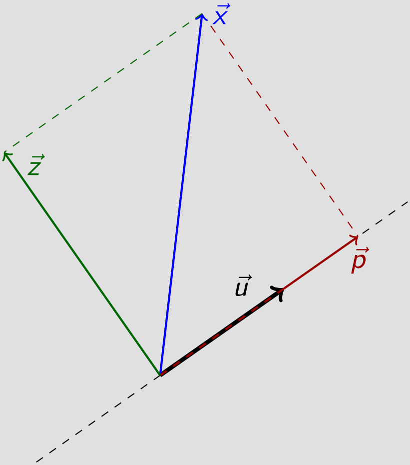
The Gram-Schmidt Orthogonalization Procedure

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Orthogonal Projection Onto a Vector

Let $\vec{u}$ be a fixed vector, and $\vec{x}$ a variable vector.



The *orthogonal projection of $\vec{x}$ onto $\vec{u}$* is the pictured vector $\vec{p}$ which is parallel to $\vec{u}$ (so, $\vec{p} = s\vec{u}$ for some scalar) and has the property that $\vec{z} = \vec{x} - \vec{p} \perp \vec{u}$.

For this to hold, we need $\vec{z} \cdot \vec{u} = 0$, which allows us to determine s. We find that

$$s = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

and thus

$$\boxed{\vec{p} = \text{Proj}_{\vec{u}}(\vec{x}) = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\,\vec{u}}.$$

Orthogonal Projection onto a Vector Subspace $\mathbb{W}$

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an orthog basis for a vector subspace $\mathbb{W}$ of $\mathbb{R}^n$.

Theorem (Orthogonal Decomposition Theorem)

Each vector $\vec{x}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\vec{x} = \vec{p} + \vec{z} \quad \text{where } \vec{p} \text{ is in } \mathbb{W} \text{ and } \vec{z} \text{ is in } \mathbb{W}^\perp.$$

In fact,

$$\vec{p} = \sum_{i=1}^{k} \mathrm{Proj}_{\vec{b}_i}(\vec{x}) = \sum_{i=1}^{k} \frac{\vec{x} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i}\, \vec{b}_i \quad \text{and } \vec{z} = \vec{x} - \vec{p}.$$

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We call $\vec{p}$ the *orthogonal projection of $\vec{x}$ onto $\mathbb{W}$*, and write $\vec{p} = \mathrm{Proj}_{\mathbb{W}}(\vec{x})$.

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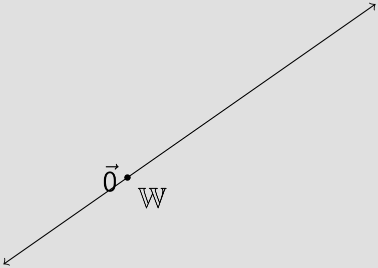

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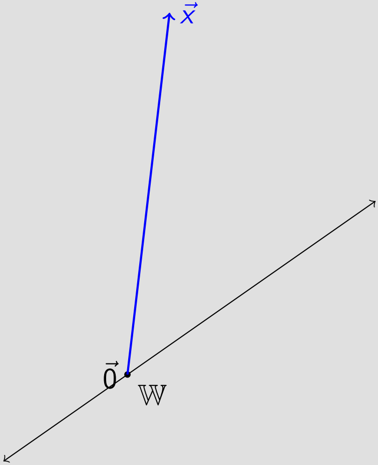

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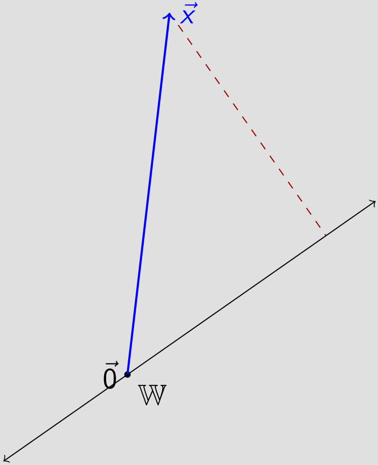

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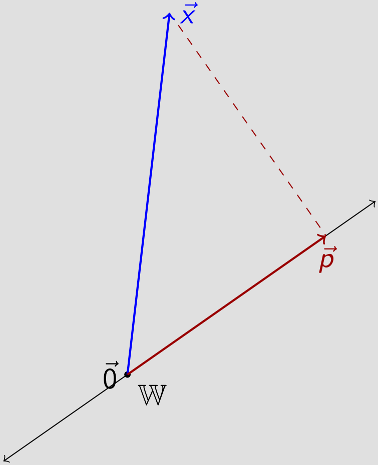

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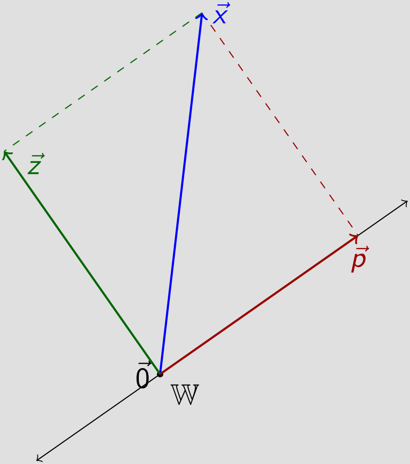

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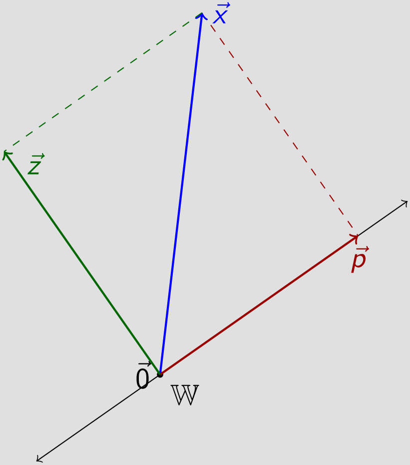

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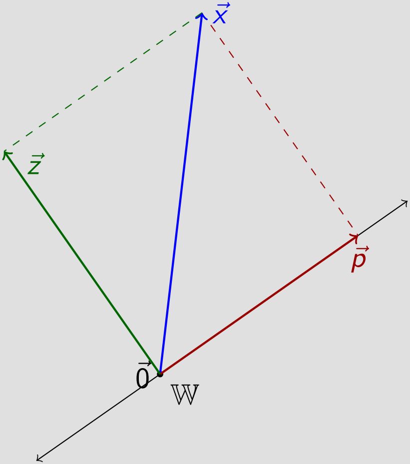

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Given a basis, how do we get an orthogonal basis?

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How do we get $\vec{v}_2, \ldots, \vec{v}_k$?

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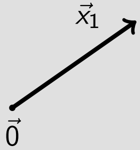

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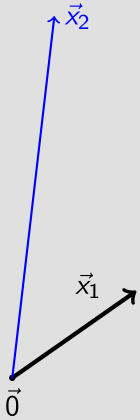

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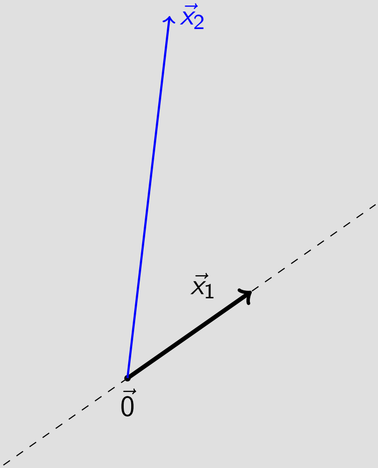

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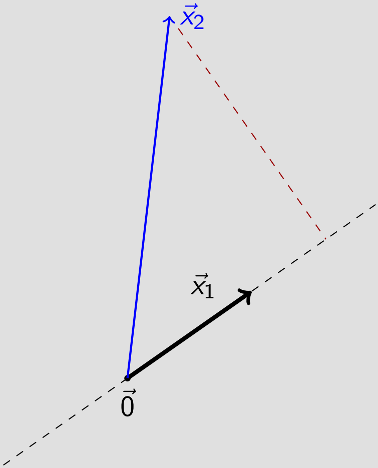

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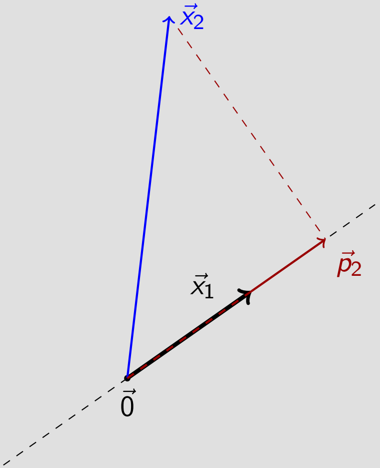

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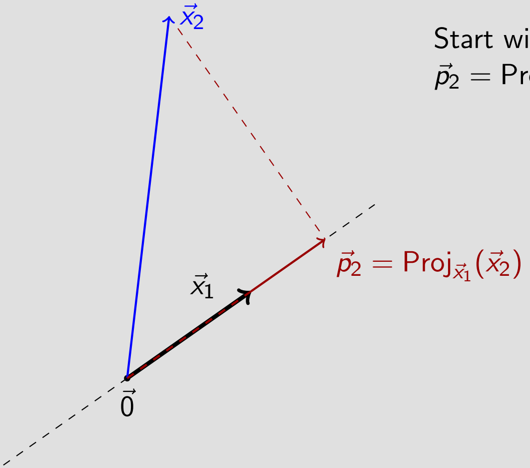

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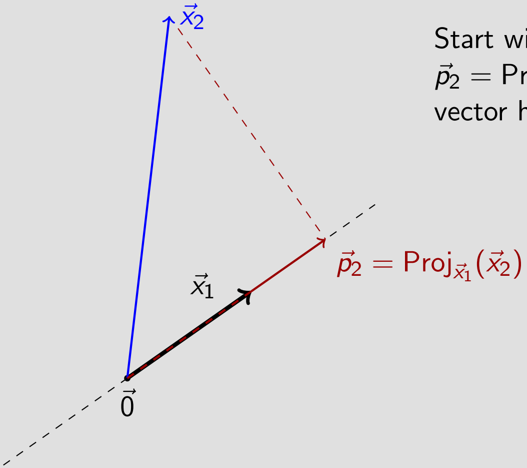

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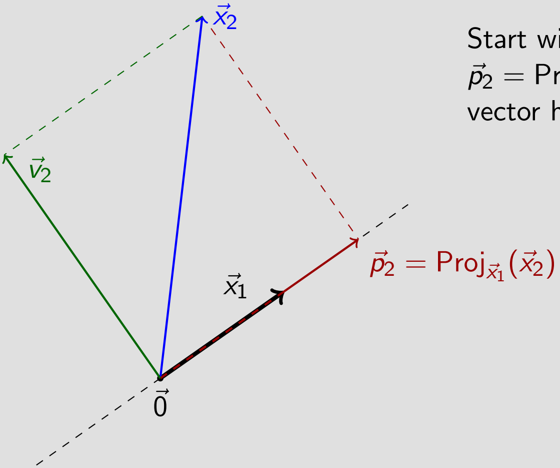

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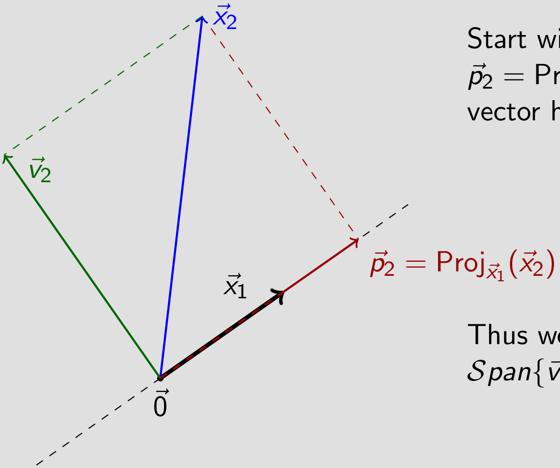

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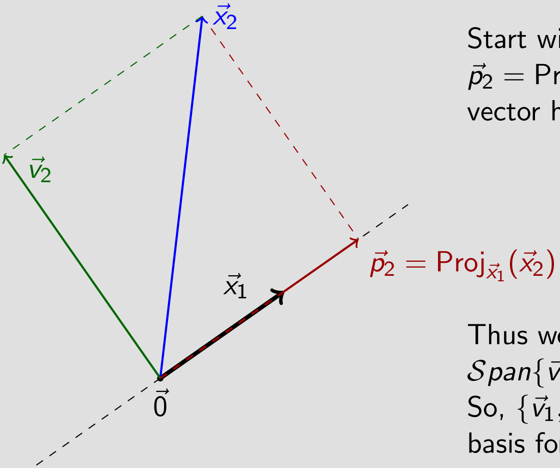

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Already have $\vec{v}_1 = \vec{x}_1$ and $\vec{v}_2 = \vec{x}_2 - \vec{p}_2$ where $\vec{p}_2 = \text{Proj}_{\vec{x}_1}(\vec{x}_2)$.

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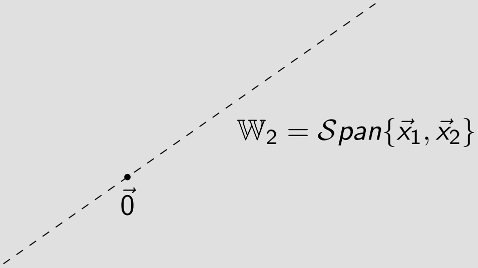
Already have $\vec{v}_1 = \vec{x}_1$ and $\vec{v}_2 = \vec{x}_2 - \vec{p}_2$ where $\vec{p}_2 = \mathrm{Proj}_{\vec{x}_1}(\vec{x}_2)$. Also know that $\mathbb{W}_2 = \mathcal{S}pan\{\vec{x}_1, \vec{x}_2\} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2\}$.



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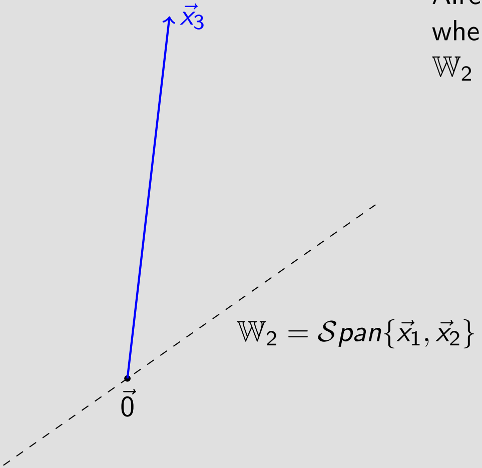

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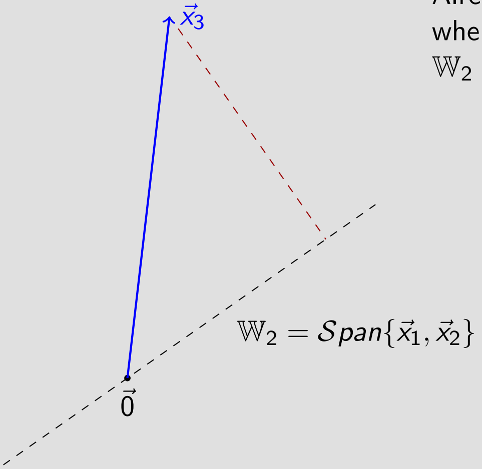

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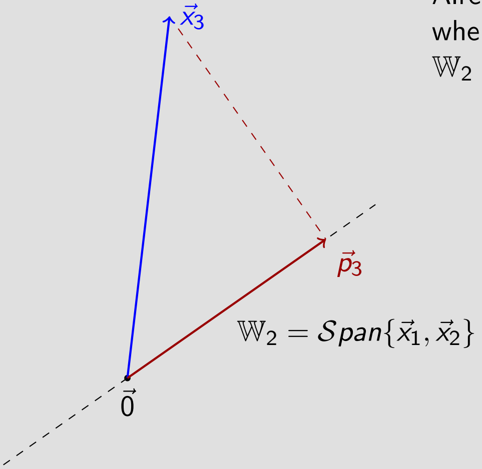

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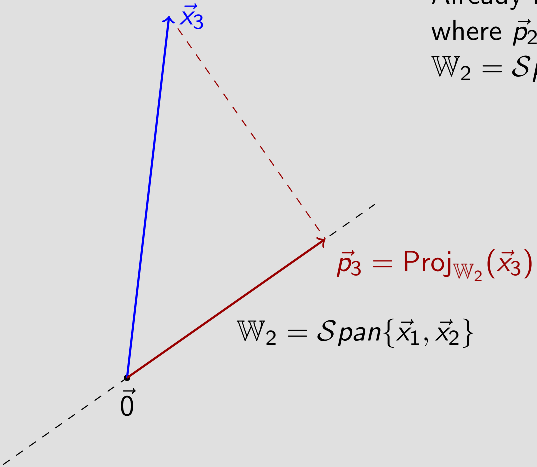

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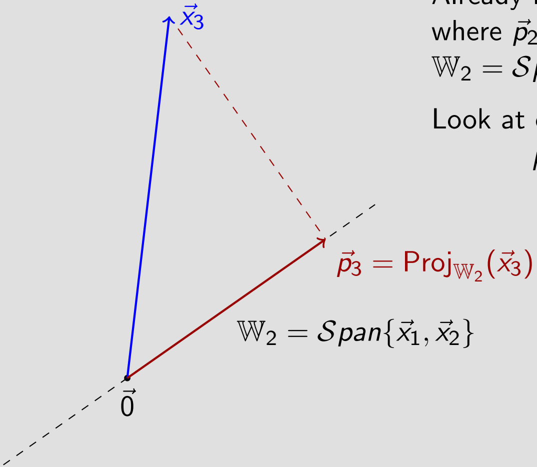

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Look at orthogonal projection

$$\vec{p}_3 = \text{Proj}_{\mathbb{W}_2}(\vec{x}_3) \text{ of } \vec{x}_3 \text{ onto } \mathbb{W}_2.$$

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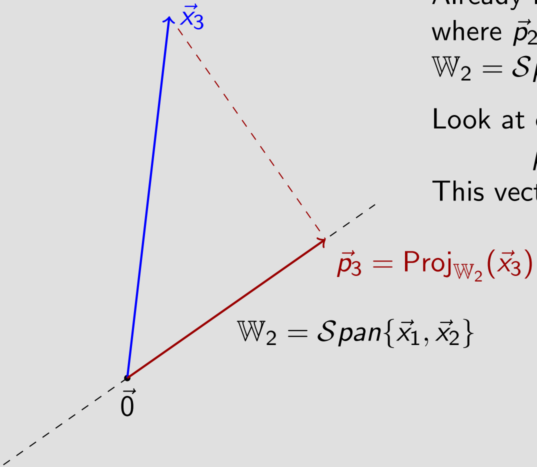

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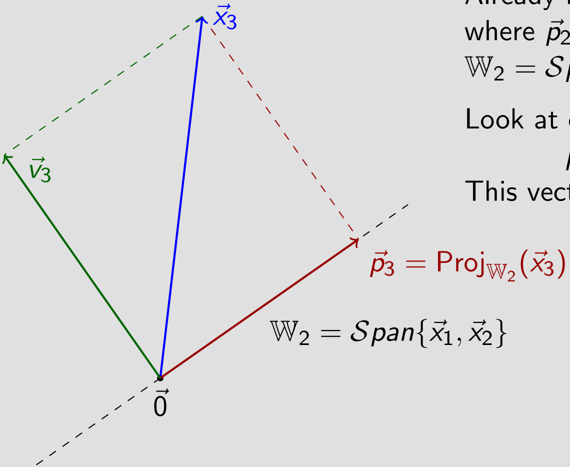

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$$\vec{v}_3 = \vec{x}_3 - \vec{p}_3 \perp \mathbb{W}_2.$$

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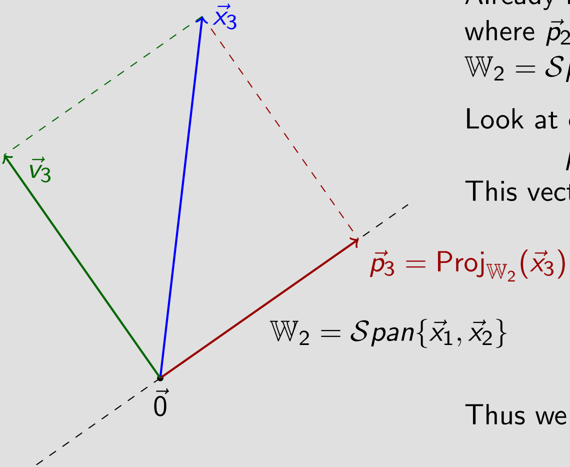

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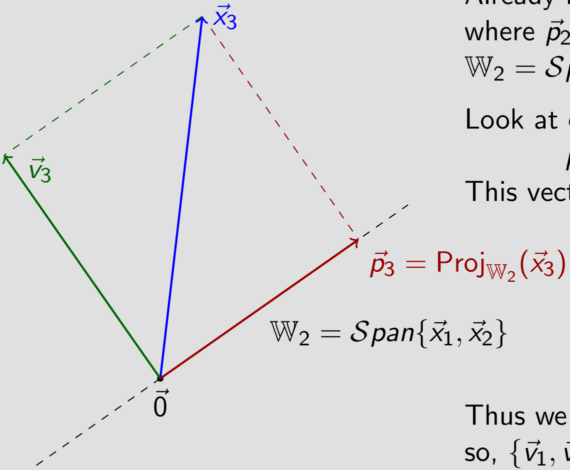

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This vector has the property that

$$\vec{v}_3 = \vec{x}_3 - \vec{p}_3 \perp \mathbb{W}_2.$$

Thus we find that $\vec{v}_1 \perp \vec{v}_2 \perp \vec{v}_3 \perp \vec{v}_1$, so, $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is the desired orthogonal basis for $\mathbb{W}$.

Start with $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

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$\vec{v}_2 = \vec{x}_2 - \vec{x}_1 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}$. Finally, $\vec{v}_3 = \vec{x}_3 - \vec{p}_3$ where $\vec{p}_3 = \mathrm{Proj}_{\mathbb{W}_2}(\vec{x}_3)$.

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Thus $\vec{p}_3 = \vec{x}_1$, so

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Thus $\vec{p}_3 = \vec{x}_1$, so

$\vec{v}_3 = \vec{x}_3 - \vec{x}_1 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Finally, check that

$\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$

are orthogonal.

Started with basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ where $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Started with basis $\{\vec{x}_1, \vec{x}_2, \vec{x}_3\}$ where $\vec{x}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} 2 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 1 \end{bmatrix}$.

Got orthogonal basis $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ where $\vec{v}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \vec{v}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

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To get orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, just *normalize*!

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To get orthonormal basis $\{\vec{u}_1, \vec{u}_2, \vec{u}_3\}$, just *normalize*!

Get $\vec{u}_1 = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{u}_2 = \frac{1}{\sqrt{3}} \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \vec{u}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

Start with $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}.$

Start with $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$.

First, $\vec{v}_1 = \vec{x}_1$.

Start with $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. First, $\vec{v}_1 = \vec{x}_1$.

Next, $\vec{v}_2 = \vec{x}_2 - \vec{p}_2$ where

Start with $\vec{x}_1 = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{x}_3 = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. First, $\vec{v}_1 = \vec{x}_1$.

Next, $\vec{v}_2 = \vec{x}_2 - \vec{p}_2$ where $\vec{p}_2 = \text{Proj}_{\vec{v}_1}(\vec{x}_2) = \dfrac{\vec{x}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\, \vec{v}_1 = \dfrac{2}{3}\vec{v}_1 = \dfrac{2}{3}\vec{x}_1$, so