Orthogonal Projection Example

Linear Algebra
MATH 2076

Orthogonal Projection onto a Vector Subspace $\mathbb{W}$

Let $\mathbb{W} = \mathcal{S}pan\{\vec{w}_1, \vec{w}_2\}$ where $\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$.

- We give an explicit formula for $\text{Proj}_{\mathbb{W}}(\vec{x})$ where $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \end{bmatrix}$.
- We use our formula to find $\text{Proj}_{\mathbb{W}}(\vec{v})$ when $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$.
- Then we find the distance d from $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}$ to $\mathbb{W}$.

Orthogonal Projection Onto $\mathbb{W} = \mathcal{S}pan\{\vec{w}_1, \vec{w}_2\}$

Recall that the LT $\mathbb{R}^n \xrightarrow{\text{Proj}_{\mathbb{W}}} \mathbb{R}^n$ (orthogonal projection onto $\mathbb{W}$) is given by

$$\text{Proj}_{\mathbb{W}}(\vec{x}) = \sum_{i=1}^{k} \frac{\vec{x} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i}\, \vec{b}_i\,;$$

this requires that $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an orthog basis for $\mathbb{W}$.
Alternatively, if $\mathcal{U} = \{\vec{u}_1, \vec{u}_2, \ldots, \vec{u}_k\}$ is an *orthon* basis for $\mathbb{W}$, then

$$\boxed{\text{Proj}_{\mathbb{W}}(\vec{x}) = P\vec{x}} \text{ where } P = \sum_{i=1}^{k} \vec{u}_i\, \vec{u}_i^{\,T} = U\,U^T \text{ and } U = \begin{bmatrix}\vec{u}_1 & \vec{u}_2 & \cdots & \vec{u}_k\end{bmatrix}.$$

$\mathbb{W} = \mathcal{S}pan\{\vec{w}_1, \vec{w}_2\}$ and

$$\vec{w}_1 = \begin{bmatrix}1\\0\\1\\0\end{bmatrix}, \vec{w}_2 = \begin{bmatrix}1\\1\\1\\1\end{bmatrix}.$$

Evidently, $\mathcal{B} = \{\vec{w}_1, \vec{w}_2\}$ is a basis for $\mathbb{W}$, but $\mathcal{B}$ is **not** orthogonal! So what now?

We write $\vec{w}_2 = \vec{p} + \vec{v}_2$ where $\vec{p} \parallel \vec{w}_1 \perp \vec{v}_2$.
Then $\{\vec{w}_1, \vec{v}_2\}$ is an *orthogonal* basis for $\mathbb{W}$!

Have $\mathbb{W} = \mathcal{S}pan\{\vec{w}_1, \vec{w}_2\}$; $\vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$. Put $\vec{v}_1 = \vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}$.

Want $\vec{w}_2 = \vec{p} + \vec{v}_2$ where $\vec{p} \parallel \vec{v}_1 \perp \vec{v}_2$. Take

$$\vec{p} = \text{Proj}_{\vec{v}_1}(\vec{w}_2) = \frac{\vec{w}_2 \cdot \vec{v}_1}{\vec{v}_1 \cdot \vec{v}_1}\,\vec{v}_1 = \vec{v}_1,$$

so

$$\vec{v}_2 = \vec{w}_2 - \vec{p} = \vec{w}_2 - \vec{v}_1 = \vec{w}_2 - \vec{w}_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}.$$

Clearly $\{\vec{v}_1, \vec{v}_2\}$ is an *orthogonal* basis for $\mathbb{W}$! Even better, setting $\vec{u}_i = \dfrac{\vec{v}_i}{|\vec{v}_i|}$, get an *orthonormal* basis $\mathcal{U} = \{\vec{u}_1, \vec{u}_2\}$ for $\mathbb{W}$. Then, $P = UU^T = \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix} \begin{bmatrix} \vec{u}_1 & \vec{u}_2 \end{bmatrix}^T$ is the standard matrix for the LT $\text{Proj}_{\mathbb{W}}$.

Have *orthonormal* basis $\mathcal{U} = \{\vec{u}_1, \vec{u}_2\}$ for $\mathbb{W} = \mathcal{S}pan\{\vec{w}_1, \vec{w}_2\}$ where

$$\vec{u}_i = \frac{\vec{v}_i}{|\vec{v}_i|} = \frac{1}{\sqrt{2}}\,\vec{v}_i\,.$$

Then,

$$P = UU^T = \begin{bmatrix}\vec{u}_1 & \vec{u}_2\end{bmatrix}\begin{bmatrix}\vec{u}_1 & \vec{u}_2\end{bmatrix}^T = \frac{1}{2}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}$$

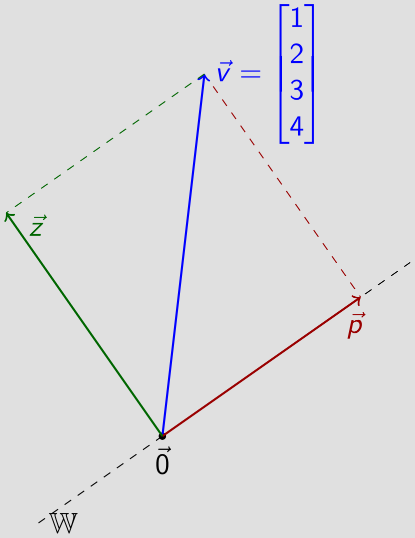
is the standard matrix for the linear transformation $\text{Proj}_{\mathbb{W}}$. Thus

$$\text{Proj}_{\mathbb{W}}(\vec{x}) = P\vec{x} = \frac{1}{2}\begin{bmatrix}1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\\ 1 & 0 & 1 & 0\\ 0 & 1 & 0 & 1\end{bmatrix}\begin{bmatrix}x_1\\ x_2\\ x_3\\ x_4\end{bmatrix} = \frac{1}{2}\begin{bmatrix}x_1 + x_3\\ x_2 + x_4\\ x_1 + x_3\\ x_2 + x_4\end{bmatrix}.$$

So, $\vec{p} = \text{Proj}_{\mathbb{W}}\left(\begin{bmatrix}1\\ 2\\ 3\\ 4\end{bmatrix}\right) = \begin{bmatrix}2\\ 3\\ 2\\ 3\end{bmatrix}$.

$$\vec{v}_1 = \begin{bmatrix}1\\ 0\\ 1\\ 0\end{bmatrix}, \vec{v}_2 = \begin{bmatrix}0\\ 1\\ 0\\ 1\end{bmatrix}$$

Finding the Distance d from $\vec{v}$ to the Subspace $\mathbb{W}$



The *orthogonal projection of* $\vec{v}$ *onto* $\mathbb{W}$ is the pictured vector $\vec{p}$ which lies in $\mathbb{W}$ and has the property that

$$\vec{z} = \vec{v} - \vec{p} \perp \mathbb{W}.$$

Since

$$\vec{p} = \begin{bmatrix} 2 \\ 3 \\ 2 \\ 3 \end{bmatrix} \text{ we see that } \vec{z} = \begin{bmatrix} -1 \\ 1 \\ 1 \\ 1 \end{bmatrix}$$

so $d = \text{dist}(\vec{v}, \mathbb{W}) = \|\vec{z}\| = 2.$