Orthogonal Projection onto a Vector Subspace of $\mathbb{R}^n$

Linear Algebra
MATH 2076

Orthogonal Complement

Definition (Orthogonal Complement of a Vector Subspace)

The *orthogonal complement* of a vector subspace $\mathbb{W}$ in $\mathbb{R}^n$ is

$$\mathbb{W}^\perp = \left\{\text{all } \vec{x} \text{ in } \mathbb{R}^n \text{ with } \vec{w} \perp \vec{x} \text{ for all } \vec{w} \text{ in } \mathbb{W}\right\}.$$

$\mathbb{W}^\perp$ is a vector subspace of $\mathbb{R}^n$, with $\dim \mathbb{W}^\perp = n - \dim \mathbb{W}$

Also, $\mathbb{R}^n = \mathbb{W} \oplus \mathbb{W}^\perp$ which means that every vector $\vec{x}$ in $\mathbb{R}^n$ can be written as a sum

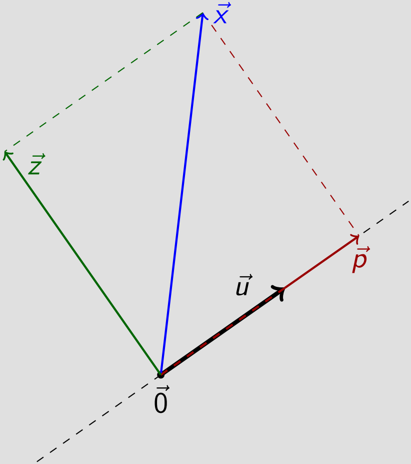
$$\vec{x} = \vec{w} + \vec{z} \quad \text{where } \vec{w} \text{ is in } \mathbb{W} \text{ and } \vec{z} \text{ is in } \mathbb{W}^\perp.$$

Here $\vec{w}$ is the *orthogonal projection of $\vec{x}$ onto $\mathbb{W}$* (the 'part' of $\vec{x}$ that is parallel to $\mathbb{W}$) and $\vec{z}$ is the *orthogonal projection of $\vec{x}$ onto $\mathbb{W}^\perp$* (the 'part' of $\vec{x}$ that is orthogonal to $\mathbb{W}$).

How do we find $\vec{w}$ and $\vec{z}$?

Orthogonal Projection Onto a Vector

Let $\vec{u}$ be a fixed vector, and $\vec{x}$ a variable vector.



The *orthogonal projection of $\vec{x}$ onto $\vec{u}$* is the pictured vector $\vec{p}$ which is parallel to $\vec{u}$ (so, $\vec{p} = s\vec{u}$ for some scalar) and has the property that $\vec{z} = \vec{x} - \vec{p} \perp \vec{u}$.
For this to hold, we need $\vec{z} \cdot \vec{u} = 0$, which allows us to determine s. We find that

$$s = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

and thus

$$\vec{p} = \mathrm{Proj}_{\vec{u}}(\vec{x}) = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\,\vec{u}.$$

Note that $\vec{x} = \vec{p} + \vec{z}$ where $\vec{p} \parallel \vec{u}$ and $\vec{z} \perp \vec{u}$.

Orthogonal Projection onto a Vector Subspace $\mathbb{W}$

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an orthog basis for a vector subspace $\mathbb{W}$ of $\mathbb{R}^n$.

Theorem (Orthogonal Decomposition Theorem)

Each vector $\vec{x}$ in $\mathbb{R}^n$ can be written uniquely in the form

$$\vec{x} = \vec{p} + \vec{z} \quad \textit{where } \vec{p} \textit{ is in } \mathbb{W} \textit{ and } \vec{z} \textit{ is in } \mathbb{W}^{\perp}.$$

In fact,

$$\vec{p} = \sum_{i=1}^{k} \mathrm{Proj}_{\vec{b}_i}(\vec{x}) = \sum_{i=1}^{k} \frac{\vec{x} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i}\, \vec{b}_i \quad \textit{and } \vec{z} = \vec{x} - \vec{p}.$$

Definition

We call $\vec{p}$ the *orthogonal projection of $\vec{x}$ onto $\mathbb{W}$*, and write $\vec{p} = \mathrm{Proj}_{\mathbb{W}}(\vec{x})$.

$\vec{p} = \mathrm{Proj}_{\mathbb{W}}(\vec{x})$ is the vector in $\mathbb{W}$ that is nearest to $\mathbb{W}$, and so

$$\mathrm{dist}(\vec{x}, \mathbb{W}) = \|\vec{x} - \vec{p}\| = \|\vec{z}\|.$$

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$ and $\mathbb{W} = \mathcal{S}pan\{\vec{w}_1, \vec{w}_2\}$ where $\vec{w}_1 = \begin{bmatrix} 2 \\ 5 \\ -1 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} -2 \\ 1 \\ 1 \end{bmatrix}$.
Write $\vec{x} = \vec{p} + \vec{z}$ with $\vec{p}$ in $\mathbb{W}$ and $\vec{z} \perp \mathbb{W}$, and find $\text{dist}(\vec{x}, \mathbb{W})$.

Since $\vec{w}_1 \cdot \vec{w}_2 = 0$, $\{\vec{w}_1, \vec{w}_2\}$ is an orthog basis for $\mathbb{W}$. Therefore, $\vec{p} = \text{Proj}_{\mathbb{W}}(\vec{x}) = \vec{p}_1 + \vec{p}_2$ where $\vec{p}_i = \text{Proj}_{\vec{w}_i}(\vec{x})$ for $i = 1, 2$. We compute

$$\vec{p}_1 = \text{Proj}_{\vec{w}_1}(\vec{x}) = \frac{\vec{x} \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\,\vec{w}_1 = \frac{9}{30}\vec{w}_1 = \frac{3}{10}\vec{w}_1$$

and

$$\vec{p}_2 = \text{Proj}_{\vec{w}_2}(\vec{x}) = \frac{\vec{x} \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\,\vec{w}_2 = \frac{3}{6}\vec{w}_2 = \frac{1}{2}\vec{w}_2.$$

Thus

$$\vec{p} = \vec{p}_1 + \vec{p}_2 = \frac{3}{10}\vec{w}_1 + \frac{1}{2}\vec{w}_2 = \frac{1}{5}\begin{bmatrix} -2 \\ 10 \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{z} = \vec{x} - \vec{p} = \frac{7}{5}\begin{bmatrix} 1 \\ 0 \\ 2 \end{bmatrix}.$$

Finally $\text{dist}(\vec{x}, \mathbb{W}) = \|\vec{x} - \vec{p}\| = \|\vec{z}\| = \frac{7}{5}\sqrt{5} = \frac{7}{\sqrt{5}}$.

Let $\vec{x} = \begin{bmatrix} 1 \\ 2 \\ 3 \\ 4 \end{bmatrix}, \vec{w}_1 = \begin{bmatrix} 1 \\ 0 \\ 1 \\ 0 \end{bmatrix}, \vec{w}_2 = \begin{bmatrix} 1 \\ 1 \\ -1 \\ 0 \end{bmatrix}, \vec{w}_3 = \begin{bmatrix} 0 \\ 0 \\ 0 \\ 1 \end{bmatrix}$. Find $\text{Proj}_{\mathbb{W}}(\vec{x})$ where $\mathbb{W} = \mathcal{S}pan\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$. Since $\{\vec{w}_1, \vec{w}_2, \vec{w}_3\}$ is an orthog basis for $\mathbb{W}$, $\vec{p} = \text{Proj}_{\mathbb{W}}(\vec{x}) = \vec{p}_1 + \vec{p}_2 + \vec{p}_3$ where $\vec{p}_i = \text{Proj}_{\vec{w}_i}(\vec{x})$ for $i = 1, 2, 3$. We compute

$$\vec{p}_1 = \text{Proj}_{\vec{w}_1}(\vec{x}) = \frac{\vec{x} \cdot \vec{w}_1}{\vec{w}_1 \cdot \vec{w}_1}\, \vec{w}_1 = \frac{4}{2}\vec{w}_1 = 2\vec{w}_1,$$

$$\vec{p}_2 = \text{Proj}_{\vec{w}_2}(\vec{x}) = \frac{\vec{x} \cdot \vec{w}_2}{\vec{w}_2 \cdot \vec{w}_2}\, \vec{w}_2 = \vec{0},$$

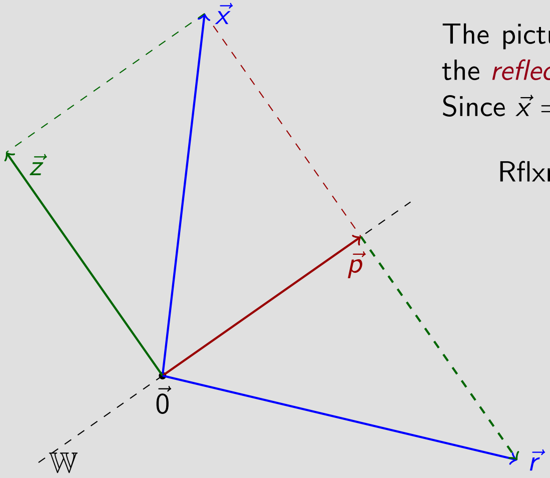
$$\vec{p}_3 = \text{Proj}_{\vec{w}_3}(\vec{x}) = \frac{\vec{x} \cdot \vec{w}_3}{\vec{w}_3 \cdot \vec{w}_3}\, \vec{w}_3 = 4\vec{w}_3.$$

Thus

$$\vec{p} = \vec{p}_1 + \vec{p}_2 + \vec{p}_3 = 2\vec{w}_1 + 4\vec{w}_3 = \begin{bmatrix} 2 \\ 0 \\ 2 \\ 4 \end{bmatrix}.$$

Orthogonal Reflection Across a Vector Subspace

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an orthog basis for a vector subspace $\mathbb{W}$ of $\mathbb{R}^n$, and $\vec{x}$ be any vector in $\mathbb{R}^n$.



The pictured vector $\vec{r} = \vec{p} - \vec{z}$ is called the *reflection of $\vec{x}$ across $\mathbb{W}$*.
Since $\vec{x} = \vec{p} + \vec{z}$, $\vec{r} = 2\vec{p} - \vec{x}$, and thus

$$\begin{aligned}\operatorname{Rflxn}_{\mathbb{W}}(\vec{x}) &= 2\operatorname{Proj}_{\mathbb{W}}(\vec{x}) - \vec{x} \\ &= \big(2\operatorname{Proj}_{\mathbb{W}} - \operatorname{Id}\big)(\vec{x}).\end{aligned}$$

Note that we need an *orthogonal basis* $\mathcal{B}$ to compute $\operatorname{Proj}_{\mathbb{W}}(\vec{x})$.