Orthogonality & Orthogonal Sets

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Definition for the Dot Product

The *dot product* of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is

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but also when $\vec{u} \neq \vec{0} \neq \vec{v}$,

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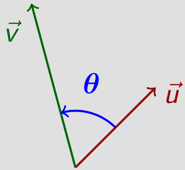

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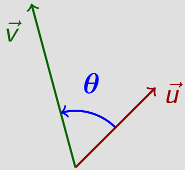

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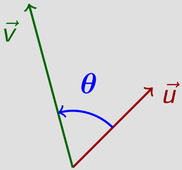

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Thus for non-zero $\vec{u}$ and $\vec{v}$, $\boxed{\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}}$.

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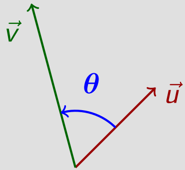

Thus for non-zero $\vec{u}$ and $\vec{v}$, $\boxed{\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}}$. Recall that $\boxed{\vec{x} \cdot \vec{x} = \|\vec{x}\|^2}$.

Properties of the Dot Product

The dot product of $\vec{u}$ and $\vec{v}$ is

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Geometric Properties

Algebraic Properties

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- $\left(\vec{u} + \vec{v}\right) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

A Useful Formula

Look at

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$$\|\vec{u}+\vec{v}\|^2 = \left(\vec{u}+\vec{v}\right)\cdot\left(\vec{u}+\vec{v}\right) = \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$

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$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 \quad \text{if and only if } \vec{u}\perp\vec{v}.$$

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$$\begin{aligned}\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) &= \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2 \\ &= \|\vec{u}\|^2 + \|\vec{v}\|^2 \quad \text{if and only if } \vec{u}\perp\vec{v}.\end{aligned}$$

The final equality above is known as *Pyhtagoras' Theorem*.

Orthogonal Complement

The *orthogonal complement* of a *non-zero* vector $\vec{a}$ in $\mathbb{R}^n$ is

$$\{\vec{a}\}^\perp = \left\{\text{all } \vec{x} \text{ in } \mathbb{R}^n \text{ with } \vec{a} \perp \vec{x}\right\}$$

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It is not hard to check that $\{\vec{a}\}^{\perp}$ is always a vector subspace of $\mathbb{R}^n$.

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Let $W = \{\vec{a}\}$ with $\vec{a} \neq \vec{0}$.

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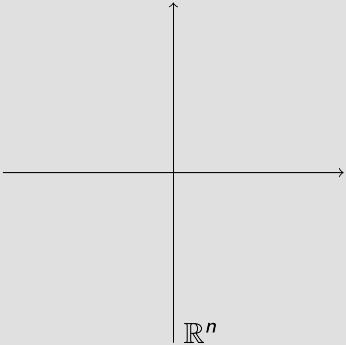

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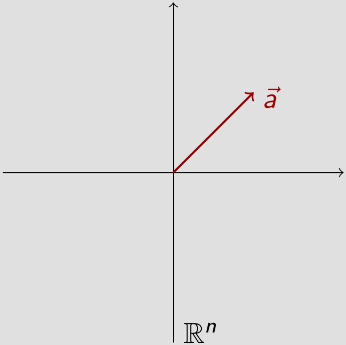

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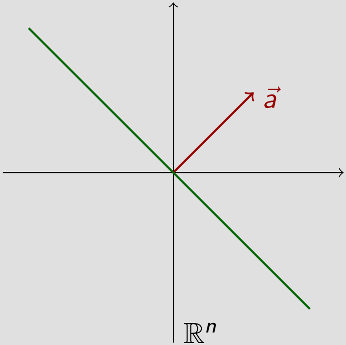

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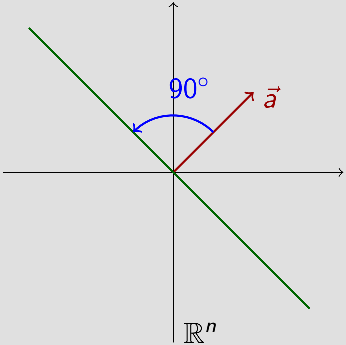

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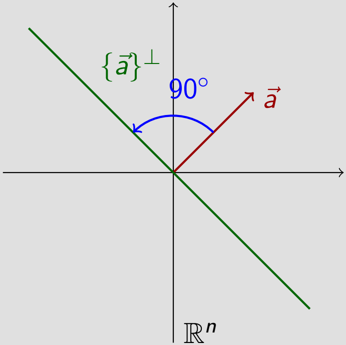

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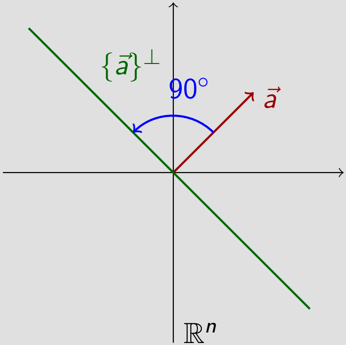

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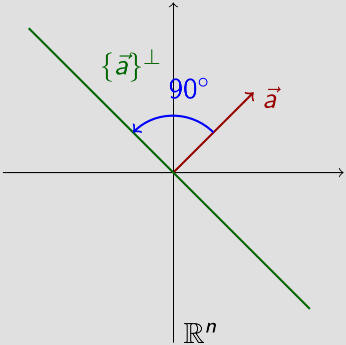
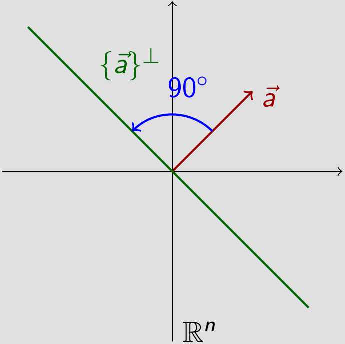


Thus we see that:

- in $\mathbb{R}^2$, $W^\perp$ is a line, Let $W = \{\vec{a}\}$ with $\vec{a} \neq \vec{0}$. Then $W^\perp = \mathcal{NS}(A)$ where $A = \vec{a}^T$.



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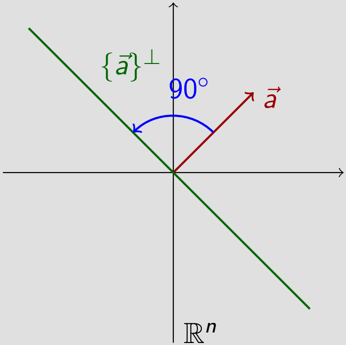

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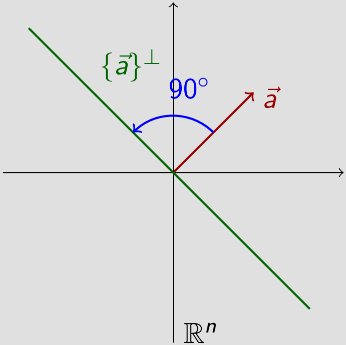

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- in $\mathbb{R}^4$, $W^\perp$ is a 3-plane,
- in $\mathbb{R}^n$, $W^\perp$ is an $(n-1)$-plane, that is, a *hyperplane*.

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The *orthogonal complement* of a non-empty set W of vectors in $\mathbb{R}^n$ is

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It is not hard to check that $W^\perp$ is always a vector subspace of $\mathbb{R}^n$.

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How do we find $\vec{w}$ and $\vec{z}$?

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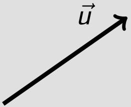

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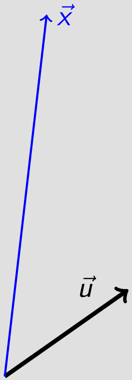

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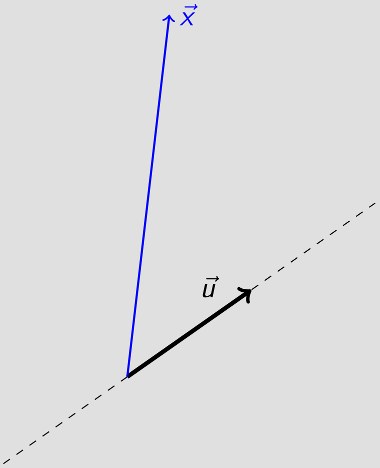

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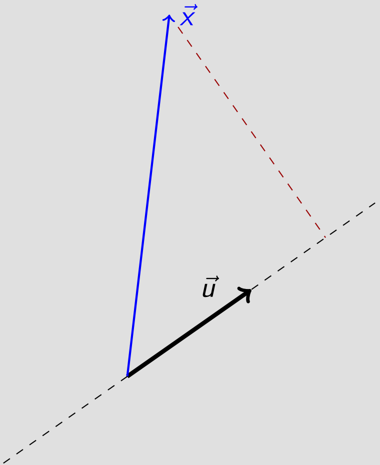

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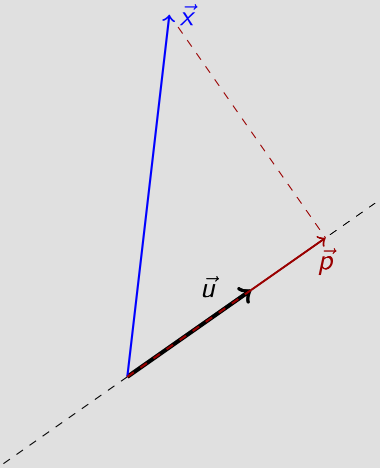

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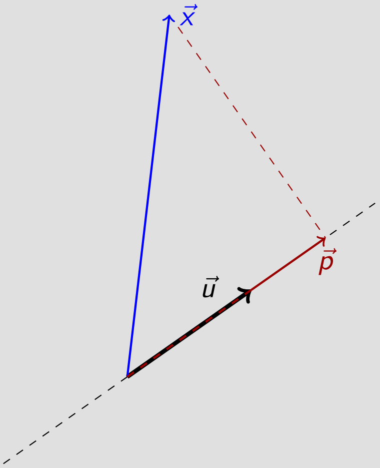

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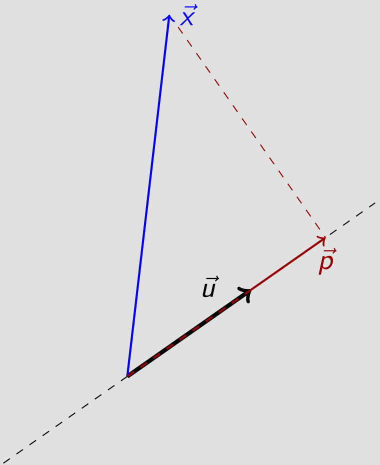

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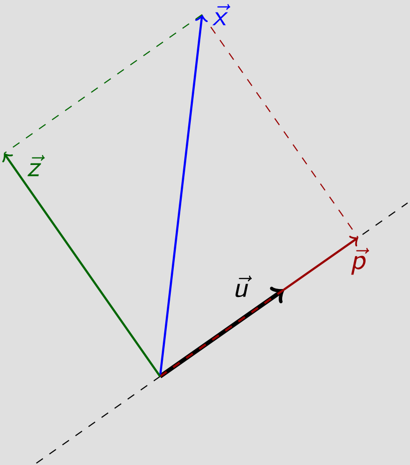

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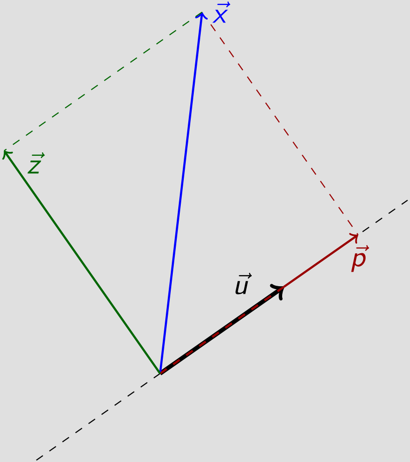

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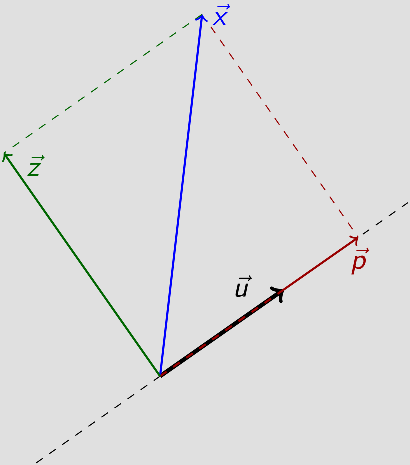

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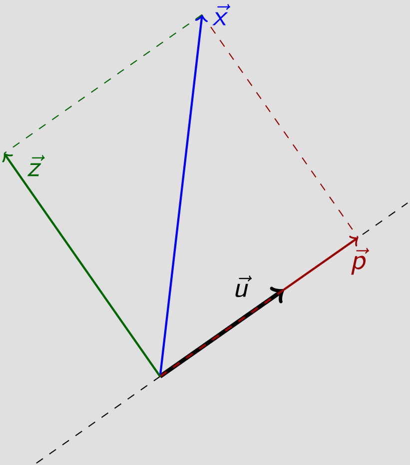

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Example of Orthogonal Projection Onto a Vector

$\vec{0}$

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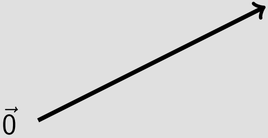

$$\vec{u} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$$

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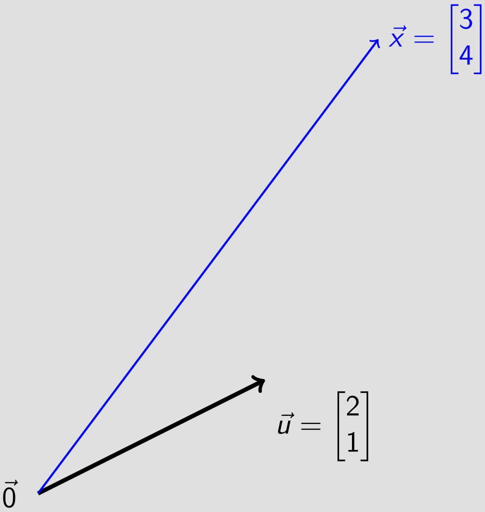

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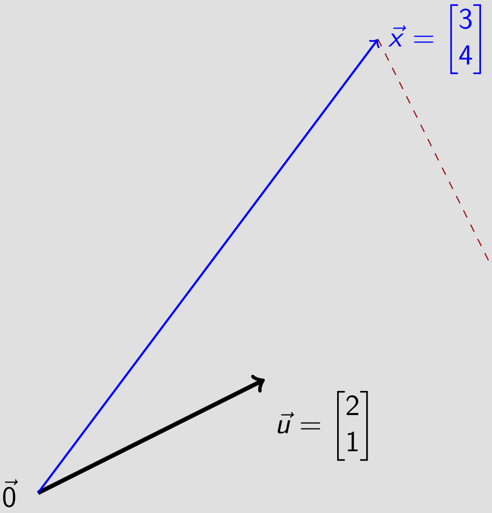

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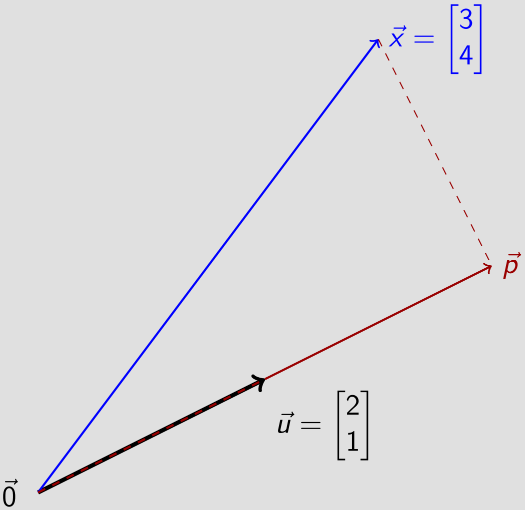

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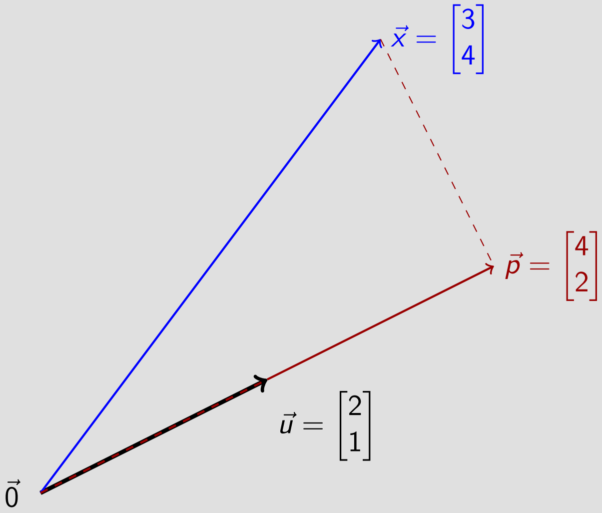

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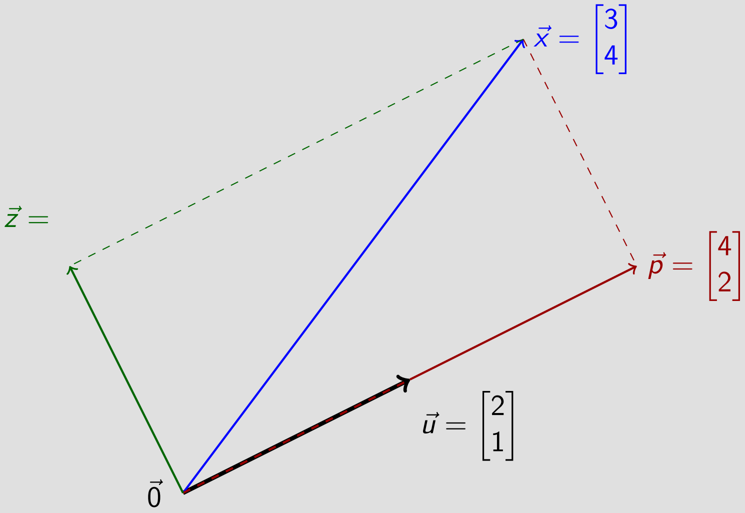

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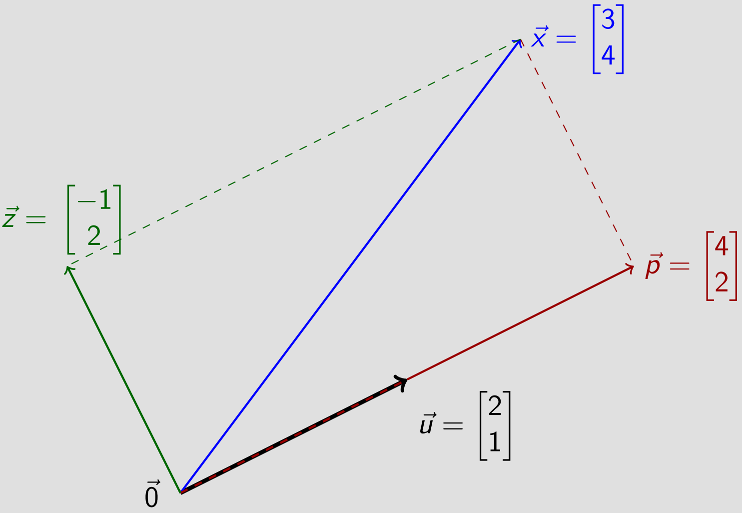

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The standard basis for $\mathbb{R}^n$ is an orthonormal basis.

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Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an ortho basis for a vector subspace $\mathbb{V}$ of $\mathbb{R}^n$.

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Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an ortho basis for a vector subspace $\mathbb{V}$ of $\mathbb{R}^n$.

For each $\vec{v}$ in $\mathbb{V}$, we can write $\vec{v} = \sum_{i=1}^{k} c_i \vec{b}_i$, where $c_1, c_2, \ldots, c_k$ are the $\mathcal{B}$-coordinates for $\vec{v}$.

Since $\mathcal{B}$ is orthogonal, it is *easy* to find these coordinates!
To find c_j, look at

$$\vec{v} \cdot \vec{b}_j = \sum_{i=1}^{k} c_i \vec{b}_i \cdot \vec{b}_j = c_j \, \vec{b}_j \cdot \vec{b}_j$$

so $c_j = \dfrac{\vec{v} \cdot \vec{b}_j}{\vec{b}_j \cdot \vec{b}_j}$. This says that

$$\vec{v} = \sum_{i=1}^{k} c_i \vec{b}_i = \sum_{i=1}^{k} \frac{\vec{v} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i} \, \vec{b}_i =$$

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