Orthogonal Projection and Orthogonal Sets

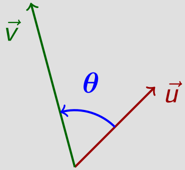
Linear Algebra
MATH 2076

Algebraic and Geometric Formulas for the Dot Product

The *dot product* of $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}$ and $\vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i = \vec{u}^T \vec{v}$$

and also when $\vec{u} \neq \vec{0} \neq \vec{v}$,



$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta$$

where θ is the angle (in $[0, \pi]$) between $\vec{u}$ and $\vec{v}$.

Thus for non-zero $\vec{u}$ and $\vec{v}$, $\boxed{\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}}$. Recall that $\boxed{\vec{x} \cdot \vec{x} = \|\vec{x}\|^2}$.

Properties of the Dot Product

The dot product of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta.$$

Geometric Properties

- $\cos\theta = \dfrac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}$
- $\vec{u} \perp \vec{v} \iff \vec{u} \cdot \vec{v} = \vec{0}$
- $\vec{u} \parallel \vec{v} \iff \vec{u} \cdot \vec{v} = \pm\|\vec{u}\| \|\vec{u}\|$
- $\vec{u} \cdot \vec{u} = \|\vec{u}\|^2$

Algebraic Properties

- $\vec{u} \cdot \vec{v}$ is a scalar
- $\vec{u} \cdot \vec{v} = \vec{v} \cdot \vec{u}$
- $(s\vec{u}) \cdot \vec{v} = \vec{u} \cdot (s\vec{v}) = s\big(\vec{u} \cdot \vec{v}\big)$
- $\big(\vec{u} + \vec{v}\big) \cdot \vec{w} = \vec{u} \cdot \vec{w} + \vec{v} \cdot \vec{w}$

Orthogonal Complement

Definition (Orthogonal Complement of a Vector Subspace)

The *orthogonal complement* of a vector subspace $\mathbb{W}$ in $\mathbb{R}^n$ is

$$\mathbb{W}^\perp = \left\{ \text{all } \vec{x} \text{ in } \mathbb{R}^n \text{ with } \vec{w} \perp \vec{x} \text{ for all } \vec{w} \text{ in } \mathbb{W} \right\}.$$

It is not hard to check that $\mathbb{W}^\perp$ is always a vector subspace of $\mathbb{R}^n$.

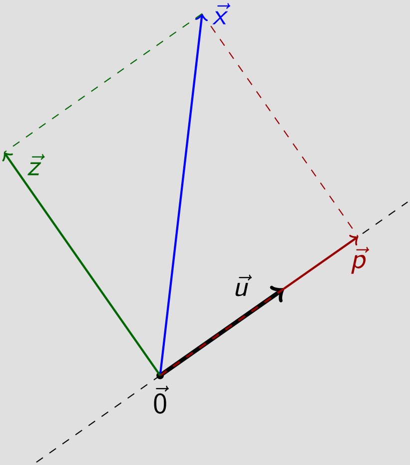
In general, if $\mathbb{W}$ is a vector subspace of $\mathbb{R}^n$, then $\mathbb{R}^n = \mathbb{W} \oplus \mathbb{W}^\perp$ and $\dim \mathbb{W}^\perp = n - \dim \mathbb{W}$. This means that every vector $\vec{x}$ in $\mathbb{R}^n$ can be written as a sum

$$\vec{x} = \vec{w} + \vec{z} \quad \text{where } \vec{w} \text{ is in } \mathbb{W} \text{ and } \vec{z} \text{ is in } \mathbb{W}^\perp.$$

Here $\vec{w}$ is the 'part' of $\vec{x}$ that is parallel to $\mathbb{W}$ and $\vec{z}$ is the 'part' of $\vec{x}$ that is orthogonal to $\mathbb{W}$. How do we find $\vec{w}$ and $\vec{z}$?

Orthogonal Projection Onto a Vector

Let $\vec{u}$ be a fixed vector, and $\vec{x}$ a variable vector.



The *orthogonal projection of $\vec{x}$ onto $\vec{u}$* is the pictured vector $\vec{p}$ which is parallel to $\vec{u}$ (so, $\vec{p} = s\vec{u}$ for some scalar) and has the property that $\vec{z} = \vec{x} - \vec{p} \perp \vec{u}$.
For this to hold, we need $\vec{z} \cdot \vec{u} = 0$, which allows us to determine s. We find that

$$s = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}$$

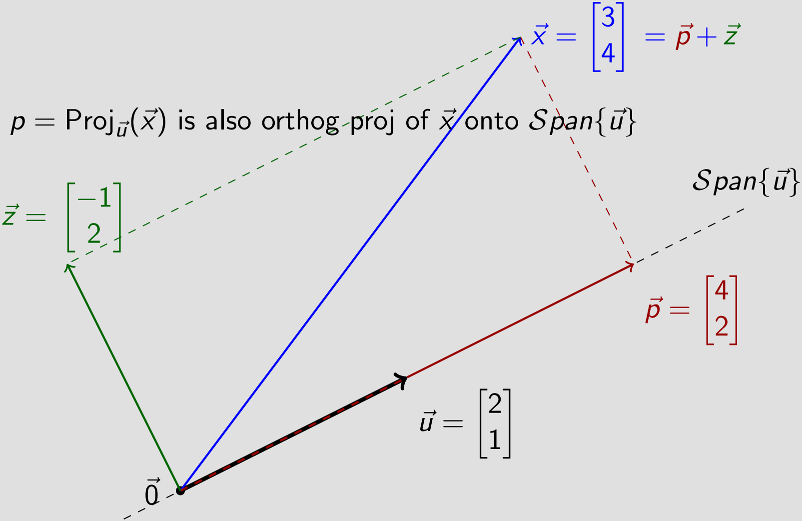
and thus

$$\vec{p} = \mathrm{Proj}_{\vec{u}}(\vec{x}) = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\,\vec{u}.$$

Note that $\vec{x} = \vec{p} + \vec{z}$ where $\vec{p} \parallel \vec{u}$ and $\vec{z} \perp \vec{u}$.

Example of Orthogonal Projection Onto a Vector

$p = \text{Proj}_{\vec{u}}(\vec{x})$ is also orthog proj of $\vec{x}$ onto $\mathcal{S}pan\{\vec{u}\}$



$$\vec{p} = \text{Proj}_{\vec{u}}(\vec{x}) = \frac{\vec{x} \cdot \vec{u}}{\vec{u} \cdot \vec{u}}\,\vec{u} = \frac{10}{5}\,\vec{u} = 2\vec{u} \text{ and } \vec{z} = \vec{x} - \vec{p} = \begin{bmatrix} 3 \\ 4 \end{bmatrix} - \begin{bmatrix} 4 \\ 2 \end{bmatrix}$$

Orthogonal Sets of Vectors

A set S of vectors (say, in $\mathbb{R}^n$) is *orthogonal* if and only if any two vectors in S are orthogonal, i.e., for all $\vec{u}, \vec{v}$ in S with $\vec{u} \neq \vec{v}$, $\vec{u} \cdot \vec{v} = 0$.

Theorem

Any set of non-zero orthogonal vectors is linearly independent.

Definition (Orthogonal and Orthonormal Bases)

Let $\mathbb{V}$ be a vector subspace of $\mathbb{R}^n$. We call $\mathcal{B}$ an *orthogonal basis* for $\mathbb{V}$ if $\mathcal{B}$ is both a basis and an orthogonal set of vectors. We call $\mathcal{U}$ an *orthonormal basis* for $\mathbb{V}$ if $\mathcal{U}$ is an orthogonal basis of *unit* vectors.

The standard basis for $\mathbb{R}^n$ is an orthonormal basis.

If $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ is an orthogonal basis, then $\vec{b}_i \cdot \vec{b}_j = 0$ when $i \neq j$.

Coordinates Relative to an Orthogonal Basis

Let $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \ldots, \vec{b}_k\}$ be an orthog basis for a vector subspace $\mathbb{V}$ of $\mathbb{R}^n$.

For each $\vec{v}$ in $\mathbb{V}$, we can write $\vec{v} = \sum_{i=1}^{k} c_i \vec{b}_i$, where $c_1, c_2, \ldots, c_k$ are the $\mathcal{B}$-coordinates for $\vec{v}$.

Since $\mathcal{B}$ is orthogonal, it is *easy* to find these coordinates!
To find c_j, look at

$$\vec{v} \cdot \vec{b}_j = \sum_{i=1}^{k} c_i \vec{b}_i \cdot \vec{b}_j = c_j \, \vec{b}_j \cdot \vec{b}_j$$

so $c_j = \dfrac{\vec{v} \cdot \vec{b}_j}{\vec{b}_j \cdot \vec{b}_j}$. This says that

$$\vec{v} = \sum_{i=1}^{k} c_i \vec{b}_i = \sum_{i=1}^{k} \frac{\vec{v} \cdot \vec{b}_i}{\vec{b}_i \cdot \vec{b}_i} \, \vec{b}_i = \sum_{i=1}^{k} \mathrm{Proj}_{\vec{b}_i}(\vec{v}). \;\ddot\smile$$