Orthogonal Complements of Null Space & Column Space

Linear Algebra
MATH 2076

Orthogonality

For $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$, $\boxed{\vec{u} \cdot \vec{v} = \vec{u}^T \vec{v} = \sum_{i=1}^n u_i v_i = \|\vec{u}\| \|\vec{v}\| \cos\theta}$.

Definition (Orthogonality)

Two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ are *orthogonal* if and only if $\vec{u} \cdot \vec{v} = 0$. When this holds, we write $\vec{u} \perp \vec{v}$.

- $\vec{0}$ is orthogonal to every other vector.
- $\vec{0}$ is the *only* vector with this property.
- If $\vec{x} \perp \vec{v}$ for every vector $\vec{v}$, then $\vec{x} = \vec{0}$.

The above is surprisingly useful. It says that if $\vec{x} \cdot \vec{v} = 0$ for every $\vec{v}$, then $\vec{x} = \vec{0}$. This is easy to see. Suppose $\vec{x}$ has the property that $\vec{x} \cdot \vec{v} = 0$ for every $\vec{v}$. Apply with $\vec{v} = \vec{x}$ to get $\vec{x} \cdot \vec{x} = 0$, which says $\|\vec{x}\| = 0$, so $\vec{x} = \vec{0}$.

Orthogonal Complements

Definition (Orthogonal Complement of a Set)

The *orthogonal complement* of a non-empty set W of vectors in $\mathbb{R}^n$ is

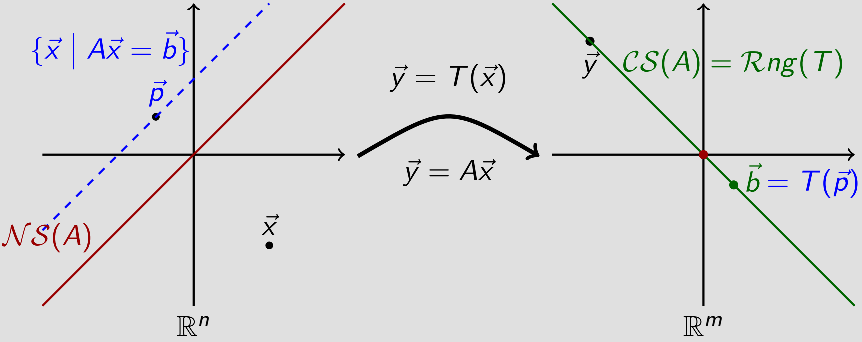
$$W^{\perp} = \left\{ \text{all } \vec{x} \text{ in } \mathbb{R}^n \text{ with } \vec{w} \perp \vec{x} \text{ for all } \vec{w} \text{ in } W \right\}.$$

It is not hard to check that $W^{\perp}$ is always a vector subspace of $\mathbb{R}^n$. Please convince yourself that this is true.

We examine this for the null space $\mathcal{NS}(A)$ and column space $\mathcal{CS}(A)$ of a matrix A.

$A = [\vec{a}_1\ \vec{a}_2\ \ldots\ \vec{a}_n]$ an $m \times n$ matrix and $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

$$\begin{aligned} \mathcal{NS}(A) &= \{\vec{x} \mid A\vec{x} = \vec{0}\} \quad \text{and} \\ \mathcal{CS}(A) &= \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_n\} \\ &= \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\} \\ &= \mathcal{R}ng(T) \end{aligned}$$



Orthogonal Complement of a Column Space

Let A be an $m \times n$ matrix. When is $\vec{y}$ in $\left(\mathcal{CS}(A)\right)^{\perp}$?

Recall that $\mathcal{CS}(A)$ consists of all vectors $A\vec{x}$ where $\vec{x}$ ranges over all of $\mathbb{R}^n$.

So, $\vec{y}$ is in $\left(\mathcal{CS}(A)\right)^{\perp}$ iff for all $\vec{x}$ in $\mathbb{R}^n$, $\vec{y} \perp A\vec{x}$, or

$$(A^T\vec{y}) \cdot \vec{x} = (A^T\vec{y})^T\vec{x} = \vec{y}^T A\vec{x} = \vec{y} \cdot (A\vec{x}) = 0;$$

but this says that $A^T\vec{y} = \vec{0}$, or equivalently, $\vec{y}$ is in $\mathcal{NS}(A^T)$.

We conclude that $\boxed{\mathcal{CS}(A)^{\perp} = \mathcal{NS}(A^T)}$.

Also, $\mathcal{CS}(A^T)^{\perp} = \mathcal{NS}(A)$. But, $(\mathbb{W}^{\perp})^{\perp} = \mathbb{W}$, so $\boxed{\mathcal{NS}(A)^{\perp} = \mathcal{CS}(A^T)}$.

The Four Fundamental Vector SubSpaces Assoc'd with A

Each $m \times n$ matrix A has four associated canonical vector subspaces.

These are:

- the null space $\mathcal{NS}(A)$ of A (a vector subspace of $\mathbb{R}^n$),
- the column space $\mathcal{CS}(A)$ of A (a vector subspace of $\mathbb{R}^m$),
- the orthogonal complement $\mathcal{CS}(A)^\perp = \mathcal{NS}(A^T)$ (a VSS of $\mathbb{R}^m$),
- the orthogonal complement $\mathcal{NS}(A)^\perp = \mathcal{CS}(A^T)$ (a VSS of $\mathbb{R}^n$).

Let's look at a picture for these four subspaces.

For an $m \times n$ matrix $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$

$$\begin{aligned} \mathcal{NS}(A) &= \{\vec{x} \mid A\vec{x} = \vec{0}\} \quad \text{and} \\ \mathcal{CS}(A) &= \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\} \\ &= \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\} \\ &= \mathcal{R}ng(T) \text{ (when } \mathbb{R}^n \xrightarrow{T} \mathbb{R}^m \text{ is } T(\vec{x}) = A\vec{x}) \end{aligned}$$