Inner Products, Length, and Orthogonality

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Algebraic Definition for Dot Product

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Notice that for the standard basis vectors in $\mathbb{R}^n$,

$$\vec{e_i} \cdot \vec{e_j} = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

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Note that $\boxed{\vec{x} \cdot \vec{x} = \|\vec{x}\|^2.}$

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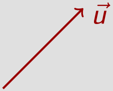

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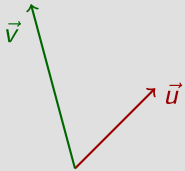

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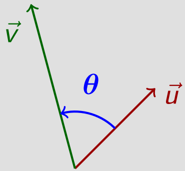

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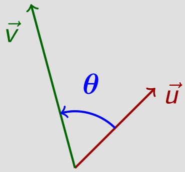

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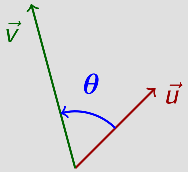

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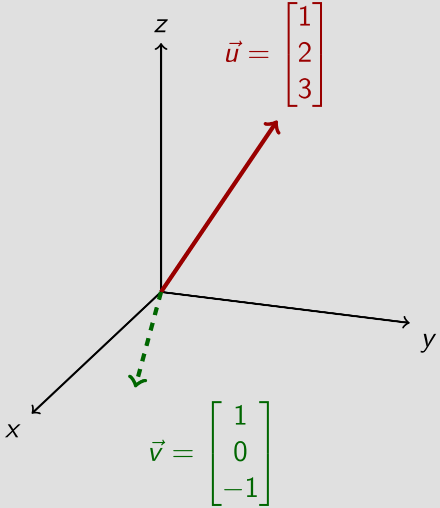
Thus for non-zero $\vec{u}$ and $\vec{v}$, $\boxed{\cos \theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}}$.

An Example



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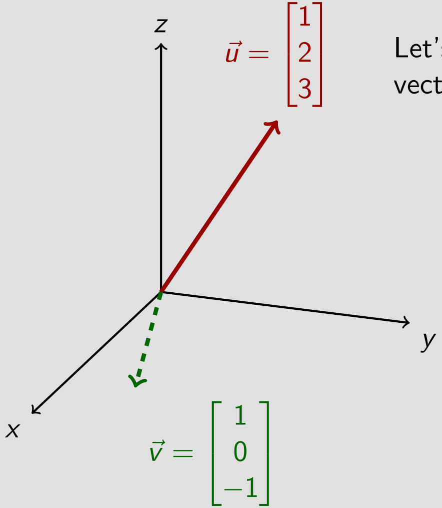

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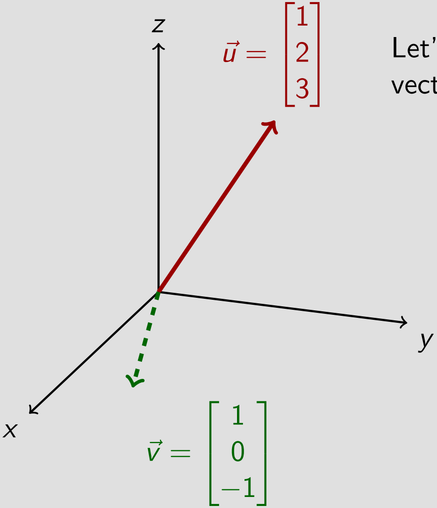

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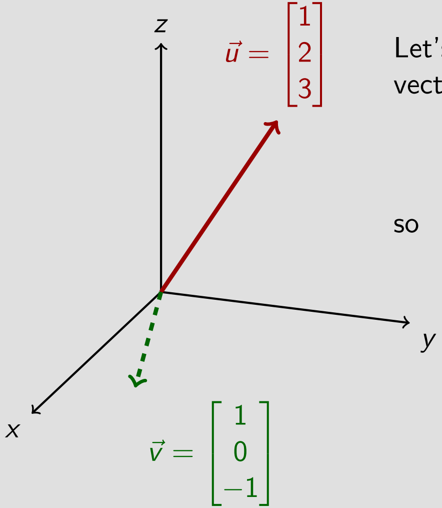

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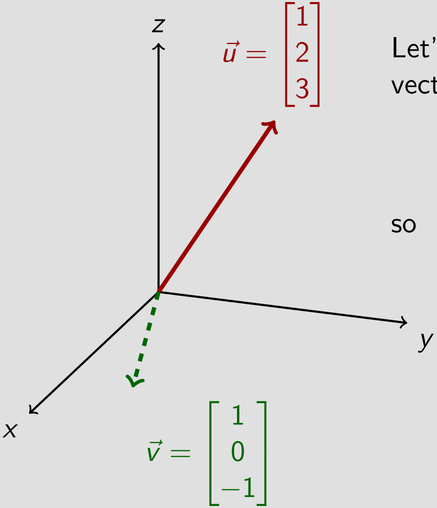

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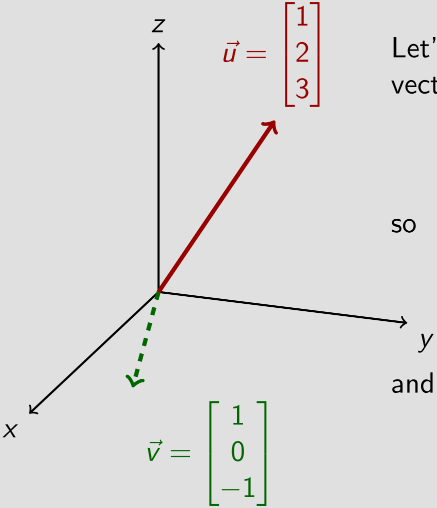

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and thus $\theta \simeq 120^\circ$.

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Find the angle between the diagonals of a cube in $\mathbb{R}^3$.

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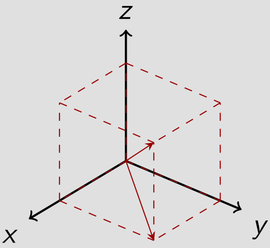

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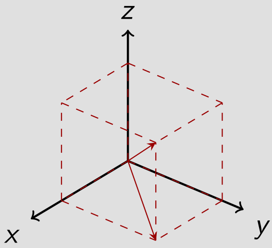

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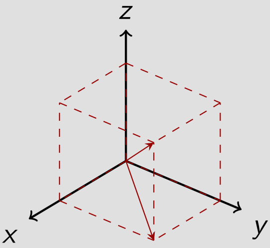

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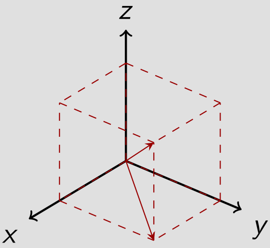

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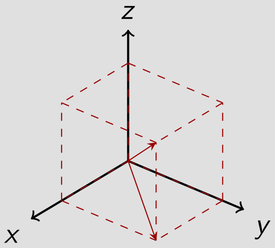

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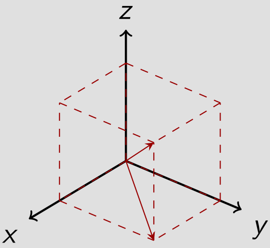

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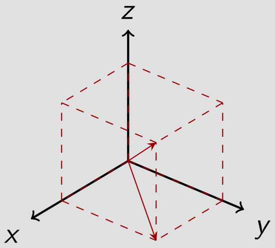

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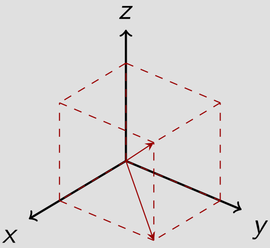

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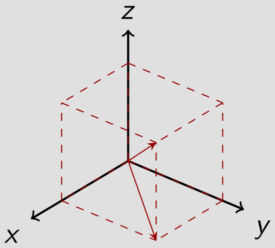

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Recall that $\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos \theta}$.

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Two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ are *orthogonal* if and only if $\vec{u} \cdot \vec{v} = 0$. When this holds, we write $\vec{u} \perp \vec{v}$.

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The final equality above is known as *Pyhtagoras' Theorem*.

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The *orthogonal complement* of a *non-zero* vector $\vec{a}$ in $\mathbb{R}^n$ is

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It is not hard to check that $W^\perp$ is always a vector subspace of $\mathbb{R}^n$.

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In general, if $\mathbb{W}$ is a vector subspace of $\mathbb{R}^n$, then $\mathbb{R}^n = \mathbb{W} \oplus \mathbb{W}^{\perp}$ and $\dim \mathbb{W}^{\perp} = n - \dim \mathbb{W}$. This means that every vector $\vec{x}$ in $\mathbb{R}^n$ can be written as a sum $\vec{x} = \vec{w} + \vec{z}$ where $\vec{w}$ is in $\mathbb{W}$ and $\vec{z}$ is in $\mathbb{W}^{\perp}$.