Inner Product, Length, and Orthogonality

Linear Algebra
MATH 2076

Algebraic Definition for Dot Product

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be vectors in $\mathbb{R}^n$. The *dot product* of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = u_1 v_1 + u_2 v_2 + \cdots + u_n v_n = \sum_{i=1}^{n} u_i v_i = \vec{u}^T \vec{v}.$$

Some Examples:

- $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix} \cdot \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix} = -2.$
- For $\vec{x}$ in $\mathbb{R}^n$, $\vec{x} \cdot \vec{e_i} = x_i$.
- For $\vec{x}$ in $\mathbb{R}^n$, $\vec{x} = \sum_{i=1}^{n} \left(\vec{x} \cdot \vec{e_i}\right) \vec{e_i}$.

Notice that for the standard basis vectors in $\mathbb{R}^n$,

$$\vec{e_i} \cdot \vec{e_j} = \delta_{ij} = \begin{cases} 0 & \text{if } i \neq j \\ 1 & \text{if } i = j. \end{cases}$$

The Length or Norm of a Vector

The *length* (or *norm* or *magnitude*) of $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$ is

$$\|\vec{x}\| = \sqrt{x_1 x_1 + x_2 x_2 + \cdots + x_n x_n} = \left(\sum_{i=1}^{n} x_i^2\right)^{1/2} = \left(\vec{x} \cdot \vec{x}\right)^{1/2}.$$

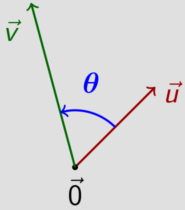
For example, if $\vec{v} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$, then $\|\vec{v}\| = \sqrt{14}$.

Note that $\boxed{\vec{x} \cdot \vec{x} = \|\vec{x}\|^2.}$

Geometric Definition for Dot Product

Let $\vec{u} = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_n \end{bmatrix}, \vec{v} = \begin{bmatrix} v_1 \\ v_2 \\ \vdots \\ v_n \end{bmatrix}$ be *non-zero* vectors in $\mathbb{R}^n$.

Let θ be the angle (in $[0, \pi]$) between $\vec{u}$ and $\vec{v}$.

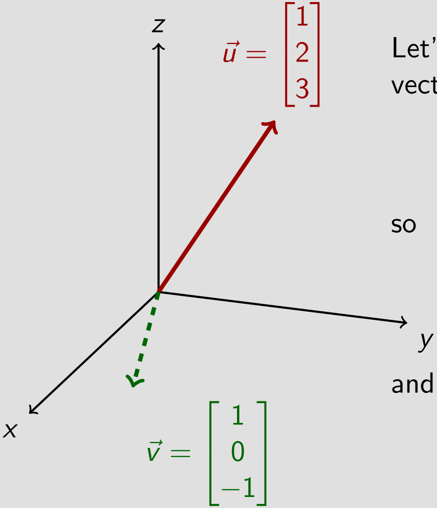


The *dot product* of $\vec{u}$ and $\vec{v}$ is

$$\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta.$$

Thus for non-zero $\vec{u}$ and $\vec{v}$, $\boxed{\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\| \|\vec{v}\|}}$.

An Example



Let's find the angle between the pictured vectors $\vec{u}, \vec{v}$. We have

$$\vec{u} \cdot \vec{v} = -2, \ \|\vec{u}\| = \sqrt{14}, \ \|\vec{v}\| = \sqrt{2}$$

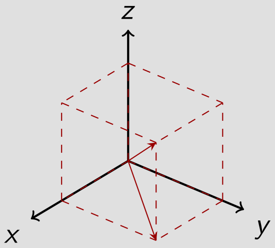
so

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{-2}{\sqrt{14}\sqrt{2}} = \frac{-1}{\sqrt{7}}$$

and thus $\theta \simeq 120^\circ$.

Another Example

Find the angle between the diagonals of a cube in $\mathbb{R}^3$.



Let $\vec{u}$ be the "main diagonal", so $\vec{u} = \vec{e}_1 + \vec{e}_2 + \vec{e}_3$.
Let $\vec{v}$ be the "floor diagonal", so $\vec{v} = \vec{e}_1 + \vec{e}_2$. Then

$$\vec{u} \cdot \vec{v} = 2, \ \|\vec{u}\| = \sqrt{3}, \ \|\vec{v}\| = \sqrt{2}$$

so

$$\cos\theta = \frac{\vec{u} \cdot \vec{v}}{\|\vec{u}\|\|\vec{v}\|} = \frac{2}{\sqrt{3}\sqrt{2}} = \sqrt{2/3}$$

and thus $\theta \simeq 35^\circ$.

Orthogonality

Recall that $\boxed{\vec{u} \cdot \vec{v} = \|\vec{u}\| \|\vec{v}\| \cos\theta}$.

Definition (Orthogonality)

Two vectors $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ are *orthogonal* if and only if $\vec{u} \cdot \vec{v} = 0$. When this holds, we write $\vec{u} \perp \vec{v}$.

Note that:

- $\vec{0}$ is orthogonal to every other vector.
- $\vec{0}$ is the *only* vector with this property.
- If $\vec{x} \perp \vec{v}$ for every vector $\vec{v}$, then $\vec{x} = \vec{0}$.

Some simple examples:

$$\begin{bmatrix} 1 \\ 1 \end{bmatrix} \perp \begin{bmatrix} -1 \\ 1 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \end{bmatrix} \perp \begin{bmatrix} -2 \\ 1 \end{bmatrix}, \begin{bmatrix} a \\ b \end{bmatrix} \perp \begin{bmatrix} -b \\ a \end{bmatrix}.$$

A Useful Formula

Look at

$$\|\vec{u}+\vec{v}\|^2 = (\vec{u}+\vec{v})\cdot(\vec{u}+\vec{v}) = \|\vec{u}\|^2 + 2\,\vec{u}\cdot\vec{v} + \|\vec{v}\|^2$$
$$= \|\vec{u}\|^2 + \|\vec{v}\|^2 \quad \text{if and only if } \vec{u}\perp\vec{v}.$$

The final statement above is known as *Pythagoras' Theorem*.

Orthogonal Complement of $W = \{\vec{a}\}$

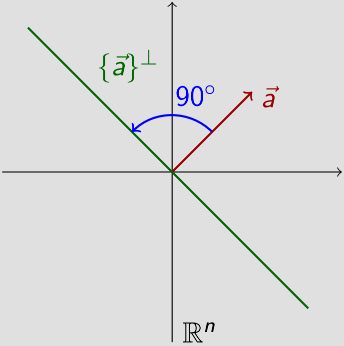
The *orthogonal complement* of a *non-zero* vector $\vec{a}$ in $\mathbb{R}^n$ is

$$\{\vec{a}\}^{\perp} = \left\{\text{all } \vec{x} \text{ in } \mathbb{R}^n \text{ with } \vec{a} \perp \vec{x}\right\} = \mathcal{NS}\left(\vec{a}^T\right).$$

It is not hard to check that $\{\vec{a}\}^{\perp}$ is always a vector subspace of $\mathbb{R}^n$.

Orthogonal Complement of $W = \{\vec{a}\}$

Let $W = \{\vec{a}\}$ with $\vec{a} \neq \vec{0}$. Then
$W^{\perp} = \left\{\text{all } \vec{x} \text{ in } \mathbb{R}^n \text{ with } \vec{a} \perp \vec{x}\right\} = \mathcal{NS}(A^T)$ where $A = \vec{a}$.



Thus we see that:

- in $\mathbb{R}^2$, $W^{\perp}$ is a line,
- in $\mathbb{R}^3$, $W^{\perp}$ is a 2-plane,
- in $\mathbb{R}^4$, $W^{\perp}$ is a 3-plane,
- in $\mathbb{R}^n$, $W^{\perp}$ is an $(n-1)$-plane, that is, a *hyperplane*.

Orthogonal Complement

The *orthogonal complement* of a *non-zero* vector $\vec{a}$ in $\mathbb{R}^n$ is

$$\{\vec{a}\}^\perp = \{\text{all } \vec{x} \text{ in } \mathbb{R}^n \text{ with } \vec{a} \perp \vec{x}\} = \mathcal{NS}(\vec{a}^T).$$

This is the *hyperplane* in $\mathbb{R}^n$ thru $\vec{0}$ with normal vector $\vec{a}$.

Definition (Orthogonal Complement of a Set)

The *orthogonal complement* of a non-empty set W of vectors in $\mathbb{R}^n$ is

$$W^\perp = \{\text{all } \vec{x} \text{ in } \mathbb{R}^n \text{ with } \vec{w} \perp \vec{x} \text{ for all } \vec{w} \text{ in } W\}.$$

It is not hard to check that $W^\perp$ is always a vector subspace of $\mathbb{R}^n$.
Please convince yourself that this is true.

Orthogonal Complement of $W = \{\vec{v}, \vec{w}\}$

Let $W = \{\vec{v}, \vec{w}\}$ with $\vec{v} \nparallel \vec{w}$. Then $W^\perp = \mathcal{NS}(A^T)$ where $A = \left[\vec{v}\ \vec{w}\right]$.

Here we see that:

- in $\mathbb{R}^3$, $W^\perp$ is a line,
- in $\mathbb{R}^4$, $W^\perp$ is a 2-plane,
- in $\mathbb{R}^n$, $W^\perp$ is an $(n-2)$-plane,

In general, if $\mathbb{W}$ is a vector subspace of $\mathbb{R}^n$, then $\mathbb{R}^n = \mathbb{W} \oplus \mathbb{W}^\perp$ and $\dim \mathbb{W}^\perp = n - \dim \mathbb{W}$. This means that every vector $\vec{x}$ in $\mathbb{R}^n$ can be written as a sum

$$\vec{x} = \vec{w} + \vec{z} \quad \text{where } \vec{w} \text{ is in } \mathbb{W} \text{ and } \vec{z} \text{ is in } \mathbb{W}^\perp.$$

Here $\vec{w}$ is the 'part' of $\vec{x}$ that is parallel to $\mathbb{W}$ and $\vec{z}$ is the 'part' of $\vec{x}$ that is orthogonal to $\mathbb{W}$. How do we find $\vec{w}$ and $\vec{z}$?

Orthogonal Complement, Column Space, and Null Space

Above we saw that if $\mathbb{W} = \mathcal{S}pan\{\vec{v}, \vec{w}\}$, then $\mathbb{W}^{\perp} = \mathcal{NS}(A^T)$ where $A = [\vec{v}\ \vec{w}]$. Here $\mathbb{W} = \mathcal{CS}(A)$.

In general, $\boxed{\mathcal{CS}(A)^{\perp} = \mathcal{NS}(A^T)}$.

Also, $\mathcal{CS}(A^T)^{\perp} = \mathcal{NS}(A)$. But, $(\mathbb{W}^{\perp})^{\perp} = \mathbb{W}$, so $\boxed{\mathcal{NS}(A)^{\perp} = \mathcal{CS}(A^T)}$.

These are the *Four Fundamental Subspaces* assoc'd to an $m \times n$ matrix A:

- its null space, $\mathcal{NS}(A)$, a subspace of $\mathbb{R}^n$;
- $\mathcal{NS}(A)^{\perp} = \mathcal{CS}(A^T)$, a subspace of $\mathbb{R}^n$;
- its column space, $\mathcal{CS}(A)$, a subspace of $\mathbb{R}^m$;
- $\mathcal{CS}(A)^{\perp} = \mathcal{NS}(A^T)$, a subspace of $\mathbb{R}^m$.