Linear Transformations and Eigenvalues

Linear Algebra
MATH 2076
Diagonalizable Matrices

An \( n \times n \) matrix \( A \) is diagonalizable if and only if there is an eigenbasis associated with \( A \). This holds if, say, \( A \) has \( n \) distinct (real) eigenvalues, because then the associated eigenvalues are LI and hence form a basis.

In general, there is an eigenbasis associated with \( A \) if and only if the dimensions of the eigenspaces for \( A \) add up to \( n \).

Suppose \( \lambda \) is an eigenvalue for \( A \). This means that \( \lambda \) is a zero for the characteristic polynomial \( p_A \) of \( A \). Therefore, we can factor \( p_A(t) = (t - \lambda)^m q(t) \) for some \( m \).

We call \( m \) the algebraic multiplicity of the eigenvalue \( \lambda \).

We always have \( 1 \leq \dim E(\lambda) \leq m \).

We call \( \dim E(\lambda) \) the geometric multiplicity of \( \lambda \).

There is an eigenbasis associated with \( A \) if and only if for every eigenvalue \( \lambda \) the geometric multiplicity of \( \lambda \) equals the algebraic multiplicity of \( \lambda \).
Diagonalizable Matrices

An $n \times n$ matrix $A$ is *diagonalizable* if and only if there is an *eigenbasis* assoc’d with $A$. 

This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an eigenbasis assoc’d with $A$ if and only if the dimensions of the eigenspaces for $A$ add up to $n$.

Suppose $\lambda$ is an eigenvalue for $A$. This means that $\lambda$ is a zero for the characteristic polynomial $p_A$ of $A$. Therefore, we can factor $p_A(t) = (t-\lambda)^m q(t)$ for some $m$.

We call $m$ the *algebraic multiplicity* of the eigenvalue $\lambda$.

We always have $1 \leq \dim E(\lambda) \leq m$.

We call $\dim E(\lambda)$ the *geometric multiplicity* of $\lambda$.

There is an eigenbasis assoc’d with $A$ if and only if for every eigenvalue $\lambda$ the geometric multiplicity of $\lambda$ equals the algebraic multiplicity of $\lambda$. 

Section 5.4
LTs & EVs
27 March 2017 2 / 1
Diagonalizable Matrices

An $n \times n$ matrix $A$ is *diagonalizable* if and only if there is an *eigenbasis* assoc’d with $A$. This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because
Diagonalizable Matrices

An $n \times n$ matrix $A$ is *diagonalizable* if and only if there is an *eigenbasis* assoc’d with $A$. This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.
Diagonalizable Matrices

An $n \times n$ matrix $A$ is *diagonalizable* if and only if there is an *eigenbasis* assoc’d with $A$. This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an *eigenbasis* assoc’d with $A$ if and only if the dimensions of the eigenspaces for $A$ add up to $n$. 

Suppose $\lambda$ is an eigenvalue for $A$. This means that $\lambda$ is a zero for the characteristic polynomial $p_A$ of $A$. Therefore, we can factor $p_A(t)$ as $p_A(t) = (t - \lambda)^m q(t)$ for some $m$. We call $m$ the *algebraic multiplicity* of the eigenvalue $\lambda$. We always have $1 \leq \dim E(\lambda) \leq m$. We call $\dim E(\lambda)$ the *geometric multiplicity* of $\lambda$. There is an *eigenbasis* assoc’d with $A$ if and only if for every eigenvalue $\lambda$ the geometric multiplicity of $\lambda$ equals the algebraic multiplicity of $\lambda$. 

Section 5.4
LTs & EVs 27 March 2017
Diagonalizable Matrices

An $n \times n$ matrix $A$ is diagonalizable if and only if there is an eigenbasis assoc’d with $A$. This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an eigenbasis assoc’d with $A$ if and only if the dimensions of the eigenspaces for $A$ add up to $n$.

Suppose $\lambda$ is an eigenvalue for $A$. This means that
Diagonalizable Matrices

An \( n \times n \) matrix \( A \) is *diagonalizable* if and only if there is an *eigenbasis* assoc’d with \( A \). This holds if, say, \( A \) has \( n \) distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an *eigenbasis* assoc’d with \( A \) if and only if the dimensions of the eigenspaces for \( A \) add up to \( n \).

Suppose \( \lambda \) is an eigenvalue for \( A \). This means that \( \lambda \) is a zero for the characteristic polynomial \( p_A \) of \( A \).
Diagonalizable Matrices

An $n \times n$ matrix $A$ is *diagonalizable* if and only if there is an *eigenbasis* assoc’d with $A$. This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an *eigenbasis* assoc’d with $A$ if and only if the dimensions of the eigenspaces for $A$ add up to $n$.

Suppose $\lambda$ is an eigenvalue for $A$. This means that $\lambda$ is a zero for the characteristic polynomial $p_A$ of $A$. Therefore, we can factor $p_A$ as
Diagonalizable Matrices

An $n \times n$ matrix $A$ is *diagonalizable* if and only if there is an *eigenbasis* assoc’d with $A$. This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an *eigenbasis* assoc’d with $A$ if and only if the dimensions of the eigenspaces for $A$ add up to $n$.

Suppose $\lambda$ is an eigenvalue for $A$. This means that $\lambda$ is a zero for the characteristic polynomial $p_A$ of $A$. Therefore, we can factor $p_A$ as

$$ p(t) = (t - \lambda)^m q(t) \quad \text{for some } m. $$
An $n \times n$ matrix $A$ is diagonalizable if and only if there is an eigenbasis assoc’d with $A$. This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an eigenbasis assoc’d with $A$ if and only if the dimensions of the eigenspaces for $A$ add up to $n$.

Suppose $\lambda$ is an eigenvalue for $A$. This means that $\lambda$ is a zero for the characteristic polynomial $p_A$ of $A$. Therefore, we can factor $p_A$ as

$$p(t) = (t - \lambda)^m q(t) \text{ for some } m.$$ 

We call $m$ the algebraic multiplicity of the eigenvalue $\lambda$. 

Diagonalizable Matrices

An $n \times n$ matrix $A$ is **diagonalizable** if and only if there is an **eigenbasis** assoc’d with $A$. This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an **eigenbasis** assoc’d with $A$ if and only if the dimensions of the eigenspaces for $A$ add up to $n$.

Suppose $\lambda$ is an eigenvalue for $A$. This means that $\lambda$ is a zero for the characteristic polynomial $p_A$ of $A$. Therefore, we can factor $p_A$ as

$$p(t) = (t - \lambda)^m q(t)$$

for some $m$.

We call $m$ the **algebraic multiplicity** of the eigenvalue $\lambda$. We always have $1 \leq \text{dim } E(\lambda) \leq m$. 
Diagonalizable Matrices

An $n \times n$ matrix $A$ is *diagonalizable* if and only if there is an *eigenbasis* assoc’d with $A$. This holds if, say, $A$ has $n$ distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an *eigenbasis* assoc’d with $A$ if and only if the dimensions of the eigenspaces for $A$ add up to $n$.

Suppose $\lambda$ is an eigenvalue for $A$. This means that $\lambda$ is a zero for the characteristic polynomial $p_A$ of $A$. Therefore, we can factor $p_A$ as

$$p(t) = (t - \lambda)^m q(t)$$

for some $m$.

We call $m$ the *algebraic multiplicity* of the eigenvalue $\lambda$. We always have $1 \leq \dim E(\lambda) \leq m$. We call $\dim E(\lambda)$ the *geometric multiplicity* of $\lambda$. 

Diagonalizable Matrices

An \( n \times n \) matrix \( A \) is diagonalizable if and only if there is an eigenbasis assoc’d with \( A \). This holds if, say, \( A \) has \( n \) distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an eigenbasis assoc’d with \( A \) if and only if the dimensions of the eigenspaces for \( A \) add up to \( n \).

Suppose \( \lambda \) is an eigenvalue for \( A \). This means that \( \lambda \) is a zero for the characteristic polynomial \( p_A \) of \( A \). Therefore, we can factor \( p_A \) as

\[
p(t) = (t - \lambda)^m q(t) \quad \text{for some } m.
\]

We call \( m \) the algebraic multiplicity of the eigenvalue \( \lambda \). We always have \( 1 \leq \dim E(\lambda) \leq m \). We call \( \dim E(\lambda) \) the geometric multiplicity of \( \lambda \).

There is an eigenbasis assoc’d with \( A \) if and only if for every eigenvalue \( \lambda \)
Diagonalizable Matrices

An \( n \times n \) matrix \( A \) is \textit{diagonalizable} if and only if there is an \textit{eigenbasis} assoc’d with \( A \). This holds if, say, \( A \) has \( n \) distinct (real) eigenvalues, because then the assoc’d eigenvalues are LI and hence form a basis.

In general, there is an \textit{eigenbasis} assoc’d with \( A \) if and only if the dimensions of the eigenspaces for \( A \) add up to \( n \).

Suppose \( \lambda \) is an eigenvalue for \( A \). This means that \( \lambda \) is a zero for the characteristic polynomial \( p_A \) of \( A \). Therefore, we can factor \( p_A \) as

\[
p(t) = (t - \lambda)^m q(t) \quad \text{for some } m.
\]

We call \( m \) the \textit{algebraic multiplicity} of the eigenvalue \( \lambda \). We always have \( 1 \leq \text{dim } \mathbb{E}(\lambda) \leq m \). We call \( \text{dim } \mathbb{E}(\lambda) \) the \textit{geometric multiplicity} of \( \lambda \).

There is an \textit{eigenbasis} assoc’d with \( A \) if and only if for every eigenvalue \( \lambda \) the geometric multiplicity of \( \lambda \) equals the algebraic multiplicity of \( \lambda \).
Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e.,
Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. 

Suppose $V \xrightarrow{T} W$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $B$, $A$ be bases for $V$, $W$ resp.

Consider the coordinate maps $V[\cdot]_B \rightarrow \mathbb{R}^n$ and $W[\cdot]_A \rightarrow \mathbb{R}^m$. Given $\vec{v}$ in $V$, we get $[\vec{v}]_B$ in $\mathbb{R}^n$, and given $\vec{w}$ in $W$, we get $[\vec{w}]_A$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_B$, $\vec{y} = [\vec{w}]_A$.

Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

$$[T(\vec{v})]_A = [\vec{w}]_A = \vec{y} = A[\vec{v}]_B.$$ 

We call $A$ the matrix for $T$ relative to $B$ and $A$, and we write $[T]_{AB} = A$, so $[T(\vec{v})]_A = [T]_{AB}[\vec{v}]_B$. 

---

Section 5.4
LTs & EVs
27 March 2017
Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$. 

Suppose $V \xrightarrow{T} W$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $B, A$ be bases for $V, W$ resp. Consider the coordinate maps $V[\cdot]_B \rightarrow \mathbb{R}^n$ and $W[\cdot]_A \rightarrow \mathbb{R}^m$. Given $\vec{v}$ in $V$, we get $[\vec{v}]_B$ in $\mathbb{R}^n$, and given $\vec{w}$ in $W$, we get $[\vec{w}]_A$ in $\mathbb{R}^m$. Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_B$, $\vec{y} = [\vec{w}]_A$.

Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

$$T(\vec{v}) A = [\vec{w}]_A = \vec{y} = A[\vec{v}]_B. $$

We call $A$ the matrix for $T$ relative to $B$ and $A$, and we write $[T]_{AB} = A$, so $[T(\vec{v})]_A = [T]_{AB}[\vec{v}]_B$. 

Section 5.4

LTs & EVs
The Matrix of a Linear Transformation

Recall that every LT \( \mathbb{R}^n \rightarrow \mathbb{R}^m \) is a matrix transformation; i.e., there is an \( m \times n \) matrix \( A \) so that \( T(\vec{x}) = A\vec{x} \). In fact, \( \text{Col}_j(A) = T(\vec{e}_j) \).

Suppose \( \mathbb{V} \rightarrow \mathbb{W} \) is a LT. Can we view \( T \) as a matrix transformation?
Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, Col$_j(A) = T(\vec{e}_j)$.

Suppose $\mathbb{V} \xrightarrow{T} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors.
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $\mathcal{V} \to \mathcal{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $\mathcal{B}, \mathcal{A}$ be bases for $\mathcal{V}, \mathcal{W}$ resp.
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \overset{T}{\to} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(e_j)$.

Suppose $\mathbb{V} \overset{T}{\to} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $\mathcal{B}, \mathcal{A}$ be bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the coordinate maps $\mathbb{V} \overset{[\cdot]_{\mathcal{B}}}{\to} \mathbb{R}^n$ and $\mathbb{W} \overset{[\cdot]_{\mathcal{A}}}{\to} \mathbb{R}^m$. 

So,

$[T(\vec{v})]_{\mathcal{A}} = A[\vec{v}]_{\mathcal{B}}$. 

We call $A$ the matrix for $T$ relative to $\mathcal{B}$ and $\mathcal{A}$ and we write $[T]_{\mathcal{B}\mathcal{A}} = A$, so $[T(\vec{v})]_{\mathcal{A}} = [T]_{\mathcal{B}\mathcal{A}}[\vec{v}]_{\mathcal{B}}$. 

Section 5.4

LTs & EVs

27 March 2017
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $\mathbb{V} \xrightarrow{T} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $\mathcal{B}, \mathcal{A}$ be bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the coordinate maps $\mathbb{V} \xrightarrow{[\cdot]_\mathcal{B}} \mathbb{R}^n$ and $\mathbb{W} \xrightarrow{[\cdot]_\mathcal{A}} \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, we get $[\vec{v}]_\mathcal{B}$ in $\mathbb{R}^n$, and
The Matrix of a Linear Transformation

Recall that every LT \( \mathbb{R}^n \to \mathbb{R}^m \) is a matrix transformation; i.e., there is an \( m \times n \) matrix \( A \) so that \( T(\vec{x}) = A\vec{x} \). In fact, \( \text{Col}_j(A) = T(\vec{e}_j) \).

Suppose \( V \to W \) is a LT. Can we view \( T \) as a matrix transformation? Yes, if we use coordinate vectors. Let \( B, A \) be bases for \( V, W \) resp.

Consider the coordinate maps \( V \xrightarrow{[\cdot]_B} \mathbb{R}^n \) and \( W \xrightarrow{[\cdot]_A} \mathbb{R}^m \). Given \( \vec{v} \) in \( V \), we get \( [\vec{v}]_B \) in \( \mathbb{R}^n \), and given \( \vec{w} \) in \( W \), we get \( [\vec{w}]_A \) in \( \mathbb{R}^m \).
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $\mathbb{V} \xrightarrow{T} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $\mathcal{B}, \mathcal{A}$ be bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the coordinate maps $\mathbb{V} \xrightarrow{[]} \mathbb{R}^n$ and $\mathbb{W} \xrightarrow{[]} \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, we get $[\vec{v}]_{\mathcal{B}}$ in $\mathbb{R}^n$, and given $\vec{w}$ in $\mathbb{W}$, we get $[\vec{w}]_{\mathcal{A}}$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_{\mathcal{B}}$, $\vec{y} = [\vec{w}]_{\mathcal{A}}$. 
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $V \xrightarrow{T} W$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $B, A$ be bases for $V, W$ resp.

Consider the coordinate maps $V \xrightarrow{[\cdot]_B} \mathbb{R}^n$ and $W \xrightarrow{[\cdot]_A} \mathbb{R}^m$. Given $\vec{v}$ in $V$, we get $[\vec{v}]_B$ in $\mathbb{R}^n$, and given $\vec{w}$ in $W$, we get $[\vec{w}]_A$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_B$, $\vec{y} = [\vec{w}]_A$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation.
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \to \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $V \to W$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $\mathcal{B}, \mathcal{A}$ be bases for $V, W$ resp.

Consider the coordinate maps $V \xrightarrow{\mathcal{B}} \mathbb{R}^n$ and $W \xrightarrow{\mathcal{A}} \mathbb{R}^m$. Given $\vec{v}$ in $V$, we get $[\vec{v}]_{\mathcal{B}}$ in $\mathbb{R}^n$, and given $\vec{w}$ in $W$, we get $[\vec{w}]_{\mathcal{A}}$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_{\mathcal{B}}, \vec{y} = [\vec{w}]_{\mathcal{A}}$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$: 

$$T(\vec{v}) = A[\vec{v}]_{\mathcal{B}} = [\vec{w}]_{\mathcal{A}} = A[\vec{v}]_{\mathcal{B}}.$$
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $\mathbb{V} \xrightarrow{T} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $\mathcal{B}, \mathcal{A}$ be bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the coordinate maps $\mathbb{V} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^n$ and $\mathbb{W} \xrightarrow{[\cdot]_{\mathcal{A}}} \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, we get $[\vec{v}]_{\mathcal{B}}$ in $\mathbb{R}^n$, and given $\vec{w}$ in $\mathbb{W}$, we get $[\vec{w}]_{\mathcal{A}}$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_{\mathcal{B}}, \vec{y} = [\vec{w}]_{\mathcal{A}}$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

$$\vec{y} = A\vec{x}$$
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $\mathbb{V} \xrightarrow{T} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $\mathcal{B}, \mathcal{A}$ be bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the coordinate maps $\mathbb{V} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^n$ and $\mathbb{W} \xrightarrow{[\cdot]_{\mathcal{A}}} \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, we get $[\vec{v}]_{\mathcal{B}}$ in $\mathbb{R}^n$, and given $\vec{w}$ in $\mathbb{W}$, we get $[\vec{w}]_{\mathcal{A}}$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_{\mathcal{B}}, \vec{y} = [\vec{w}]_{\mathcal{A}}$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

$$[\vec{w}]_{\mathcal{A}} = \vec{y} = A\vec{x}$$
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $V \xrightarrow{T} W$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $B, A$ be bases for $V, W$ resp.

Consider the coordinate maps $V \xrightarrow{[]}_B \mathbb{R}^n$ and $W \xrightarrow{[]}_A \mathbb{R}^m$. Given $\vec{v}$ in $V$, we get $[\vec{v}]_B$ in $\mathbb{R}^n$, and given $\vec{w}$ in $W$, we get $[\vec{w}]_A$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_B$, $\vec{y} = [\vec{w}]_A$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

\[
[ T(\vec{v}) ]_A = [ \vec{w} ]_A = \vec{y} = A\vec{x}
\]
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $V \xrightarrow{T} W$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $B, A$ be bases for $V, W$ resp.

Consider the coordinate maps $V \xrightarrow{[]}_B \mathbb{R}^n$ and $W \xrightarrow{[]}_A \mathbb{R}^m$. Given $\vec{v}$ in $V$, we get $[\vec{v}]_B$ in $\mathbb{R}^n$, and given $\vec{w}$ in $W$, we get $[\vec{w}]_A$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_B$, $\vec{y} = [\vec{w}]_A$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

$$[T(\vec{v})]_A = [\vec{w}]_A = \vec{y} = A\vec{x} = A[\vec{v}]_B.$$
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(e_j)$.

Suppose $\mathbb{V} \xrightarrow{T} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $B, A$ be bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the coordinate maps $\mathbb{V} \xrightarrow{[\cdot]_B} \mathbb{R}^n$ and $\mathbb{W} \xrightarrow{[\cdot]_A} \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, we get $[\vec{v}]_B$ in $\mathbb{R}^n$, and given $\vec{w}$ in $\mathbb{W}$, we get $[\vec{w}]_A$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_B$, $\vec{y} = [\vec{w}]_A$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

$$[T(\vec{v})]_A = [\vec{w}]_A = \vec{y} = A\vec{x} = A[\vec{v}]_B.$$  

so, $[T(\vec{v})]_A = A[\vec{v}]_B$. 

Section 5.4  
LTs & EVs  
27 March 2017 3 / 1
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $\mathbb{V} \xrightarrow{T} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $B, A$ be bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the coordinate maps $\mathbb{V} \xrightarrow{[\cdot]_B} \mathbb{R}^n$ and $\mathbb{W} \xrightarrow{[\cdot]_A} \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, we get $[\vec{v}]_B$ in $\mathbb{R}^n$, and given $\vec{w}$ in $\mathbb{W}$, we get $[\vec{w}]_A$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_B$, $\vec{y} = [\vec{w}]_A$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

$$[T(\vec{v})]_A = [\vec{w}]_A = \vec{y} = A\vec{x} = A[\vec{v}]_B.$$  

so, $[T(\vec{v})]_A = A[\vec{v}]_B$. We call $A$ the matrix for $T$ relative to $B$ and $A$ and
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \overset{T}{\rightarrow} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $\mathbb{V} \overset{T}{\rightarrow} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $\mathcal{B}, \mathcal{A}$ be bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the coordinate maps $\mathbb{V} \overset{[\cdot]_\mathcal{B}}{\rightarrow} \mathbb{R}^n$ and $\mathbb{W} \overset{[\cdot]_\mathcal{A}}{\rightarrow} \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, we get $[\vec{v}]_\mathcal{B}$ in $\mathbb{R}^n$, and given $\vec{w}$ in $\mathbb{W}$, we get $[\vec{w}]_\mathcal{A}$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_\mathcal{B}$, $\vec{y} = [\vec{w}]_\mathcal{A}$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

$$[T(\vec{v})]_\mathcal{A} = [\vec{w}]_\mathcal{A} = \vec{y} = A\vec{x} = A[\vec{v}]_\mathcal{B}.$$  

so, $[T(\vec{v})]_\mathcal{A} = A[\vec{v}]_\mathcal{B}$. We call $A$ the matrix for $T$ relative to $\mathcal{B}$ and $\mathcal{A}$ and we write $[T]_{\mathcal{A}\mathcal{B}} = A$, so
The Matrix of a Linear Transformation

Recall that every LT $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is a matrix transformation; i.e., there is an $m \times n$ matrix $A$ so that $T(\vec{x}) = A\vec{x}$. In fact, $\text{Col}_j(A) = T(\vec{e}_j)$.

Suppose $\mathbb{V} \xrightarrow{T} \mathbb{W}$ is a LT. Can we view $T$ as a matrix transformation? Yes, if we use coordinate vectors. Let $B, A$ be bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the coordinate maps $\mathbb{V} \xrightarrow{[]}_B \mathbb{R}^n$ and $\mathbb{W} \xrightarrow{[]}_A \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, we get $[\vec{v}]_B$ in $\mathbb{R}^n$, and given $\vec{w}$ in $\mathbb{W}$, we get $[\vec{w}]_A$ in $\mathbb{R}^m$.

Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = [\vec{v}]_B$, $\vec{y} = [\vec{w}]_A$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix $A$:

$$[T(\vec{v})]_A = [\vec{w}]_A = \vec{y} = A\vec{x} = A[\vec{v}]_B.$$ 

so, $[T(\vec{v})]_A = A[\vec{v}]_B$. We call $A$ the **matrix for $T$ relative to $B$ and $A$** and we write $[T]_{AB} = A$, so $[T(\vec{v})]_A = [T]_{AB} [\vec{v}]_B$. 

We have a linear transformation $\mathcal{V} \xrightarrow{T} \mathcal{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathcal{V}, \mathcal{W}$ resp.
We have a linear transformation $V \xrightarrow{T} W$ and bases $B, A$ for $V, W$ resp.

Consider the $B$ and $A$ coord maps $V \to \mathbb{R}^n$ and $W \to \mathbb{R}^m$. 

\[
\begin{array}{c}
V & \xrightarrow{T} & W \\
\downarrow{[\cdot]_B} & & \downarrow{[\cdot]_A} \\
\mathbb{R}^n & & \mathbb{R}^m
\end{array}
\]
We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \to \mathbb{R}^n$ and $\mathbb{W} \to \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

$$T: \mathbb{V} \to \mathbb{W}$$

$$(\cdot)_{\mathcal{B}}: \mathbb{V} \to \mathbb{R}^n$$

$$(\cdot)_{\mathcal{A}}: \mathbb{W} \to \mathbb{R}^m$$

$$T(\vec{v}) = A [\cdot]_{\mathcal{A}} [\cdot]_{\mathcal{B}}$$
We have a linear transformation \( V \xrightarrow{T} W \) and bases \( B, A \) for \( V, W \) resp.

Consider the \( B \) and \( A \) coord maps \( V \rightarrow \mathbb{R}^n \) and \( W \rightarrow \mathbb{R}^m \). Given \( \vec{v} \) in \( V \), \( \vec{w} \) in \( W \), let 
\[
\vec{x} = [\vec{v}]_B \quad \text{and} \quad \vec{y} = [\vec{w}]_A.
\]
We have a linear transformation $\mathbb{V} \rightarrow \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \rightarrow \mathbb{R}^n$ and $\mathbb{W} \rightarrow \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let $\vec{x} = [\vec{v}]_\mathcal{B}$ and $\vec{y} = [\vec{w}]_\mathcal{A}$.

Consider the LT $\mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. 

$$\begin{align*}
\begin{array}{ccc}
\mathbb{V} & \overset{T}{\rightarrow} & \mathbb{W} \\
\mathbb{R}^n & \overset{[\cdot]_\mathcal{B}}{\downarrow} & \mathbb{R}^m \\
\vec{x} & \mapsto & \vec{y} \\
\end{array}
\end{align*}$$
We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \rightarrow \mathbb{R}^n$ and $\mathbb{W} \rightarrow \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

$\vec{x} = [\vec{v}]_\mathcal{B}$ and $\vec{y} = [\vec{w}]_\mathcal{A}$.

Consider the LT $\mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $\vec{x} \leftrightarrow \vec{y}$.

This is a matrix transformation, and
We have a linear transformation $V \to W$ and bases $B, A$ for $V, W$ resp.

Consider the $B$ and $A$ coord maps $V \to \mathbb{R}^n$ and $W \to \mathbb{R}^m$. Given $\vec{v}$ in $V$, $\vec{w}$ in $W$, let

$$\vec{x} = [\vec{v}]_B \quad \text{and} \quad \vec{y} = [\vec{w}]_A.$$ 

Consider the LT $\mathbb{R}^n \to \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. This is a matrix transformation, and

$$\vec{y} = A\vec{x}.$$
We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \rightarrow \mathbb{R}^n$ and $\mathbb{W} \rightarrow \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

$$\vec{x} = [\vec{v}]_\mathcal{B} \text{ and } \vec{y} = [\vec{w}]_\mathcal{A}.$$  

Consider the LT $\mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$.  

This is a matrix transformation, and

$$[\vec{w}]_\mathcal{A} = \vec{y} = A\vec{x}.$$
We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \rightarrow \mathbb{R}^n$ and $\mathbb{W} \rightarrow \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

$$\vec{x} = [\vec{v}]_{\mathcal{B}} \text{ and } \vec{y} = [\vec{w}]_{\mathcal{A}}.$$ 

Consider the LT $\mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. This is a matrix transformation, and

$$[T(\vec{v})]_{\mathcal{A}} = [\vec{w}]_{\mathcal{A}} = \vec{y} = A\vec{x}.$$
We have a linear transformation $V \xrightarrow{T} W$ and bases $B, A$ for $V, W$ resp.

Consider the $B$ and $A$ coord maps $V \rightarrow \mathbb{R}^n$ and $W \rightarrow \mathbb{R}^m$. Given $\vec{v}$ in $V$, $\vec{w}$ in $W$, let $\vec{x} = [\vec{v}]_B$ and $\vec{y} = [\vec{w}]_A$.

Consider the LT $\mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. This is a matrix transformation, and $\left[T(\vec{v})\right]_A = [\vec{w}]_A = \vec{y} = A\vec{x} = A[\vec{v}]_B$. 

Section 5.4

LTs & EVs

27 March 2017
We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \rightarrow \mathbb{R}^n$ and $\mathbb{W} \rightarrow \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

$$\vec{x} = [\vec{v}]_\mathcal{B} \quad \text{and} \quad \vec{y} = [\vec{w}]_\mathcal{A}.$$ 

Consider the LT $\mathbb{R}^n \rightarrow \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. This is a matrix transformation, and

$$[T(\vec{v})]_\mathcal{A} = [\vec{w}]_\mathcal{A} = \vec{y} = A\vec{x} = A[\vec{v}]_\mathcal{B}$$

where $A = [T]_{\mathcal{AB}}$. 

\[ \begin{array}{c}
\vec{v} \\
\mathbb{V} \\
\xrightarrow{T}
\end{array} \quad \begin{array}{c}
\vec{w} \\
\mathbb{W}
\end{array} \] 

\[ \begin{array}{c}
\vec{x} \\
\mathbb{R}^n \\
\xrightarrow{T}
\end{array} \quad \begin{array}{c}
\vec{y} \\
\mathbb{R}^m
\end{array} \]
Let $A$ be a diagonalizable $n \times n$ matrix.
Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$. 
Connection with Diagonalization

Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \overset{T}{\rightarrow} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an eigenbasis assoc’d with $A$; that is,
Connection with Diagonalization

Let $A$ be a *diagonalizable* $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an *eigenbasis* assoc’d with $A$; that is, there is a basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. 

Recall that $P = P \in B$; so, $P^{-1} = P^{-1}B$ which means that $[\vec{y}]_B = P^{-1}\vec{y}$.

Thus $[T(\vec{x})]_B = P^{-1}(A\vec{x}) = P^{-1}(PDP^{-1}\vec{x}) = D(P^{-1}\vec{x}) = D[\vec{x}]_B$. This says that $D = [T]_B = [T]_{BB}$.
Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an eigenbasis assoc’d with $A$; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then
Let $A$ be a *diagonalizable* $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an *eigenbasis* assoc’d with $A$; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$
Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an eigenbasis assoc’d with $A$; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$ 

Recall that $P = P_{\mathcal{E}\mathcal{B}}$; so,
Connection with Diagonalization

Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an eigenbasis assoc’d with $A$; that is, there is a basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$ 

Recall that $P = P_{\mathcal{E}B}$; so, $P^{-1} = P_{B\mathcal{E}}$ which means that
Connection with Diagonalization

Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an eigenbasis assoc’d with $A$; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

Recall that $P = P_{\mathcal{E}\mathcal{B}}$; so, $P^{-1} = P_{\mathcal{B}\mathcal{E}}$ which means that $[\vec{y}]_\mathcal{B} = P^{-1}\vec{y}$. 
Connection with Diagonalization

Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an eigenbasis assoc'd with $A$; that is, there is a basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$ 

Recall that $P = P_{\mathcal{E}B}$; so, $P^{-1} = P_{B\mathcal{E}}$ which means that $[\vec{y}]_B = P^{-1}\vec{y}$. Thus

$$\left[ T(\vec{x}) \right]_B = \ldots$$
Connection with Diagonalization

Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an eigenbasis assoc’d with $A$; that is, there is a basis $B = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$ 

Recall that $P = P_{\mathcal{E}\mathcal{B}}$; so, $P^{-1} = P_{\mathcal{B}\mathcal{E}}$ which means that $[\vec{y}]_B = P^{-1}\vec{y}$. Thus

$$\left[T(\vec{x})\right]_B = \left[A\vec{x}\right]_B = \left[\lambda_i \vec{v}_i\right]_B.$$
Connection with Diagonalization

Let $A$ be a *diagonalizable* $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an *eigenbasis* assoc’d with $A$; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$  

Recall that $P = P_{\mathcal{E}\mathcal{B}}$; so, $P^{-1} = P_{\mathcal{B}\mathcal{E}}$ which means that $[\vec{y}]_{\mathcal{B}} = P^{-1}\vec{y}$. Thus

$$\left[ T(\vec{x}) \right]_{\mathcal{B}} = \left[ A\vec{x} \right]_{\mathcal{B}} = P^{-1}(A\vec{x}) =$$
Connection with Diagonalization

Let $A$ be a *diagonalizable* $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an *eigenbasis* assoc’d with $A$; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1}$$

where

$$P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix}$$

and

$$D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

Recall that $P = P_{\mathcal{E}\mathcal{B}}$; so, $P^{-1} = P_{\mathcal{B}\mathcal{E}}$ which means that $[\vec{y}]_{\mathcal{B}} = P^{-1}\vec{y}$.

Thus

$$\left[ T(\vec{x}) \right]_{\mathcal{B}} = \left[ A\vec{x} \right]_{\mathcal{B}} = P^{-1}(A\vec{x}) = P^{-1}(PDP^{-1}\vec{x}) = \left[ D \right]_{\mathcal{B}} \left[ \vec{x} \right]_{\mathcal{B}}.$$
Let $A$ be a \textit{diagonalizable} $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an \textit{eigenbasis} assoc’d with $A$; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

Recall that $P = P_{\mathcal{E}\mathcal{B}}$; so, $P^{-1} = P_{\mathcal{B}\mathcal{E}}$ which means that $[\vec{y}]_{\mathcal{B}} = P^{-1}\vec{y}$. Thus

$$\left[ T(\vec{x}) \right]_{\mathcal{B}} = \left[ A\vec{x} \right]_{\mathcal{B}} = P^{-1}(A\vec{x}) = P^{-1}(PDP^{-1}\vec{x}) = D(P^{-1}\vec{x}) = \ldots$$
Connection with Diagonalization

Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an eigenbasis assoc’d with $A$; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

Recall that $P = P_{\mathcal{E}\mathcal{B}}$; so, $P^{-1} = P_{\mathcal{B}\mathcal{E}}$ which means that $[\vec{y}]_{\mathcal{B}} = P^{-1}\vec{y}$.

Thus

$$\begin{bmatrix} T(\vec{x}) \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} A\vec{x} \end{bmatrix}_{\mathcal{B}} = P^{-1}(A\vec{x}) = P^{-1}(PDP^{-1}\vec{x}) = D(P^{-1}\vec{x}) = D[\vec{x}]_{\mathcal{B}}.$$
Connection with Diagonalization

Let $A$ be a diagonalizable $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since $A$ is diagonalizable, there is an eigenbasis assoc’d with $A$; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for $A$. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots & \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \ddots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$ 

Recall that $P = P_{\mathcal{B}\mathcal{E}}$; so, $P^{-1} = P_{\mathcal{B}\mathcal{E}}$ which means that $[\vec{y}]_{\mathcal{B}} = P^{-1}\vec{y}$. Thus

$$[T(\vec{x})]_{\mathcal{B}} = [A\vec{x}]_{\mathcal{B}} = P^{-1}(A\vec{x}) = P^{-1}(PDP^{-1}\vec{x}) = D(P^{-1}\vec{x}) = D[\vec{x}]_{\mathcal{B}}.$$ 

This says that $D = [T]_{\mathcal{B}} = [T]_{\mathcal{B}\mathcal{B}}$. 

Section 5.4

LTs & EVs

27 March 2017 5 / 1
A 3 × 3 Example

The matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ has *simple* eigenvalues 3, 4, 6 with associated eigenvectors

Since $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$, $A$ is diagonalizable with $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & 1 \\ 1 & -2 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.

Here $D$ is the $B$-matrix for the LT $\vec{x} \mapsto A\vec{x}$. 

Section 5.4 LTs & EVs 27 March 2017 6 / 1
The matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ has *simple* eigenvalues 3, 4, 6 with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$. Since $B = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$, $A$ is diagonalizable with $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & -1 & 1 \\ -1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$. Here $D$ is the $B$-matrix for the LT $\vec{x} \mapsto A\vec{x}$.
A $3 \times 3$ Example

The matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ has simple eigenvalues $3, 4, 6$ with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Since $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is a basis for $\mathbb{R}^3$, $A$ is diagonalizable with $A = PDP^{-1}$ where $P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix}$ and $D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}$.
A 3 × 3 Example

The matrix \( A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix} \) has *simple* eigenvalues 3, 4, 6 with associated eigenvectors \( \vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \), \( \vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix} \), \( \vec{v}_3 = \begin{bmatrix} 1 \\ 1 \\ -2 \end{bmatrix} \).

Since \( B = \{ \vec{v}_1, \vec{v}_2, \vec{v}_3 \} \) is a basis for \( \mathbb{R}^3 \), \( A \) is diagonalizable with

\[
A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}.
\]

Here \( D \) is the \( B \)-matrix for the LT \( \vec{x} \mapsto A\vec{x} \).