Linear Transformations and Eigenvalues

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

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There is an *eigenbasis* assoc'd with A if and only if for *every* eigenvalue λ the geometric multiplicity of λ equals the algebraic multiplicity of λ.

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Take $\vec{w} = T(\vec{v})$, and let $\vec{x} = \big[\vec{v}\big]_{\mathcal{B}}$, $\vec{y} = \big[\vec{w}\big]_{\mathcal{A}}$. Thanks to linearity props of coord vectors, the map $\vec{x} \mapsto \vec{y}$ (from $\mathbb{R}^n$ to $\mathbb{R}^m$) is a linear transformation. Therefore, this is given by multiplication by some matrix A:

$$\big[T(\vec{v})\big]_{\mathcal{A}} = \big[\vec{w}\big]_{\mathcal{A}} = \vec{y} = A\vec{x} = A\big[\vec{v}\big]_{\mathcal{B}}.$$

so, $\big[T(\vec{v})\big]_{\mathcal{A}} = A\big[\vec{v}\big]_{\mathcal{B}}$. We call A the *matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$* and we write $\big[T\big]_{\mathcal{AB}} = A$, so $\boxed{\big[T(\vec{v})\big]_{\mathcal{A}} = \big[T\big]_{\mathcal{AB}}\big[\vec{v}\big]_{\mathcal{B}}}$.

Picture for the matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$

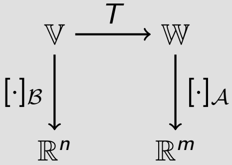
We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

$$\mathbb{V} \xrightarrow{\quad T \quad} \mathbb{W}$$

Picture for the matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$

We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

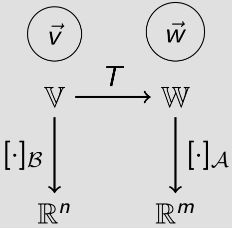
Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \to \mathbb{R}^n$ and $\mathbb{W} \to \mathbb{R}^m$.



Picture for the matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$

We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \to \mathbb{R}^n$ and $\mathbb{W} \to \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let



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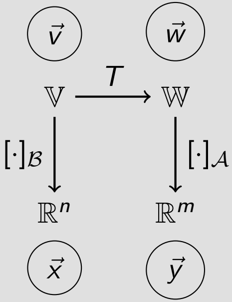
$$\vec{x} = \left[\vec{v}\right]_{\mathcal{B}} \text{ and } \vec{y} = \left[\vec{w}\right]_{\mathcal{A}}.$$



Picture for the matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$

We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \to \mathbb{R}^n$ and $\mathbb{W} \to \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

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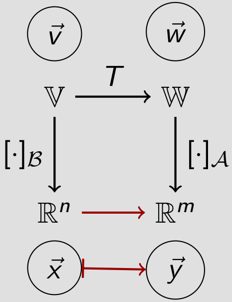
Consider the LT $\mathbb{R}^n \to \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$.



Picture for the matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$

We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

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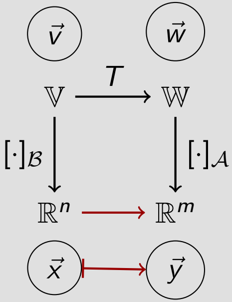
Consider the LT $\mathbb{R}^n \to \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. This is a matrix transformation, and



Picture for the matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$

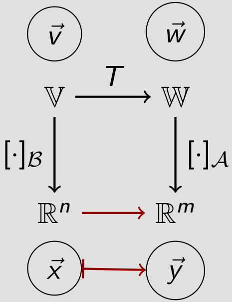
We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \to \mathbb{R}^n$ and $\mathbb{W} \to \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

$$\vec{x} = \left[\vec{v}\right]_{\mathcal{B}} \text{ and } \vec{y} = \left[\vec{w}\right]_{\mathcal{A}}.$$

Consider the LT $\mathbb{R}^n \to \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. This is a matrix transformation, and

$$\vec{y} = A\vec{x}$$



Picture for the matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$

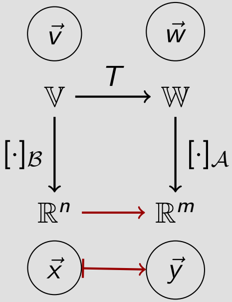
We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \to \mathbb{R}^n$ and $\mathbb{W} \to \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

$$\vec{x} = \left[\vec{v}\right]_{\mathcal{B}} \text{ and } \vec{y} = \left[\vec{w}\right]_{\mathcal{A}}.$$

Consider the LT $\mathbb{R}^n \to \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. This is a matrix transformation, and

$$\left[\vec{w}\right]_{\mathcal{A}} = \vec{y} = A\vec{x}$$



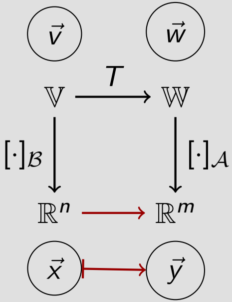
Picture for the matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$

We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \to \mathbb{R}^n$ and $\mathbb{W} \to \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

$$\vec{x} = \left[\vec{v}\right]_{\mathcal{B}} \text{ and } \vec{y} = \left[\vec{w}\right]_{\mathcal{A}}.$$

Consider the LT $\mathbb{R}^n \to \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. This is a matrix transformation, and $\left[T(\vec{v})\right]_{\mathcal{A}} = \left[\vec{w}\right]_{\mathcal{A}} = \vec{y} = A\vec{x}$



Picture for the matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$

We have a linear transformation $\mathbb{V} \xrightarrow{T} \mathbb{W}$ and bases $\mathcal{B}, \mathcal{A}$ for $\mathbb{V}, \mathbb{W}$ resp.

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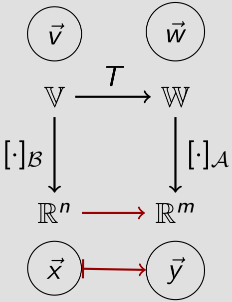

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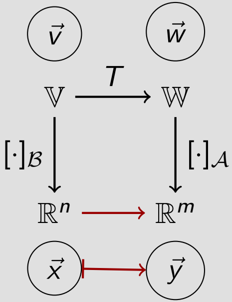
Consider the LT $\mathbb{R}^n \to \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$.
This is a matrix transformation, and
$\left[T(\vec{v})\right]_{\mathcal{A}} = \left[\vec{w}\right]_{\mathcal{A}} = \vec{y} = A\vec{x} = A\left[\vec{v}\right]_{\mathcal{B}}$
where $A = \left[T\right]_{\mathcal{AB}}$.



Connection with Diagonalization

Let A be a *diagonalizable* $n \times n$ matrix.

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$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \ldots \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

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$$\Big[T(\vec{x})\Big]_{\mathcal{B}} = \Big[A\vec{x}\Big]_{\mathcal{B}} = P^{-1}(A\vec{x}) = P^{-1}(PDP^{-1}\vec{x}) = D(P^{-1}\vec{x}) =$$

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Let A be a *diagonalizable* $n \times n$ matrix. Define $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ by $T(\vec{x}) = A\vec{x}$.

Since A is diagonalizable, there is an *eigenbasis* assoc'd with A; that is, there is a basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for A. Assume $A\vec{v}_i = \lambda_i \vec{v}_i$. Then

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \dots \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \dots & 0 \\ 0 & \lambda_2 & \dots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \dots & \lambda_n \end{bmatrix}.$$

Recall that $P = P_{\mathcal{E}\mathcal{B}}$; so, $P^{-1} = P_{\mathcal{B}\mathcal{E}}$ which means that $\left[\vec{y}\right]_{\mathcal{B}} = P^{-1}\vec{y}$. Thus

$$\left[T(\vec{x})\right]_{\mathcal{B}} = \left[A\vec{x}\right]_{\mathcal{B}} = P^{-1}(A\vec{x}) = P^{-1}(PDP^{-1}\vec{x}) = D(P^{-1}\vec{x}) = D\left[\vec{x}\right]_{\mathcal{B}}.$$

This says that $D = \left[T\right]_{\mathcal{B}} = \left[T\right]_{\mathcal{B}\mathcal{B}}$.

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Here D is the $\mathcal{B}$-matrix for the LT $\vec{x} \mapsto A\vec{x}$.