Linear Transformations and Similar Matrices

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Similar Matrices and Diagonalizable Matrices

Two $n \times n$ matrices A and B are *similar* if and only if there is an invertible matrix P such that $A = PBP^{-1}$ (and then we also have $B = P^{-1}AP = QAQ^{-1}$ where $Q = P^{-1}$).

An $n \times n$ matrix A is *diagonalizable* if and only if it is similar to a diagonal matrix; that is, there are a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

Here we want to better understand these ideas. In fact, these are statements about linear transformations.

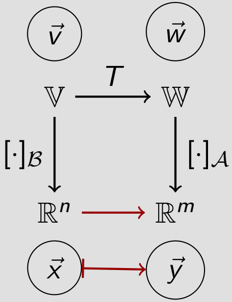
Please review Sections 1.9, 4.7 (the text and related videos) and especially the video Chpt4Sect7M4LT. This material actually has little to do with eigen stuff!

The Matrix of a Linear Transformation

Suppose $\mathbb{V} \xrightarrow{T} \mathbb{W}$ is a LT and $\mathcal{B}, \mathcal{A}$ are bases for $\mathbb{V}, \mathbb{W}$ resp.

Consider the $\mathcal{B}$ and $\mathcal{A}$ coord maps $\mathbb{V} \to \mathbb{R}^n$ and $\mathbb{W} \to \mathbb{R}^m$. Given $\vec{v}$ in $\mathbb{V}$, $\vec{w}$ in $\mathbb{W}$, let

$$\vec{x} = \big[\vec{v}\big]_{\mathcal{B}} \text{ and } \vec{y} = \big[\vec{w}\big]_{\mathcal{A}}.$$



Consider the LT $\mathbb{R}^n \to \mathbb{R}^m$ given by $\vec{x} \mapsto \vec{y}$. This is a matrix transformation, and

$$\big[T(\vec{v})\big]_{\mathcal{A}} = \big[\vec{w}\big]_{\mathcal{A}} = \vec{y} = M\vec{x} = M\big[\vec{v}\big]_{\mathcal{B}}$$

so $\big[T(\vec{v})\big]_{\mathcal{A}} = M\big[\vec{v}\big]_{\mathcal{B}}$.

We call $M = \big[T\big]_{\mathcal{A}\mathcal{B}}$ the *matrix for T relative to $\mathcal{B}$ and $\mathcal{A}$*.

Thus $\boxed{\big[T(\vec{v})\big]_{\mathcal{A}} = \big[T\big]_{\mathcal{A}\mathcal{B}}\big[\vec{v}\big]_{\mathcal{B}}}$.

The $\mathcal{B}$-Matrix for a Linear Transformation

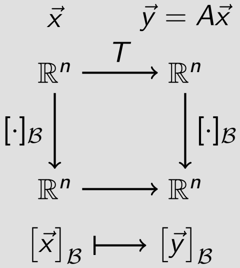
Suppose $\mathbb{V} = \mathbb{R}^n = \mathbb{W}$ and $\mathcal{B} = \mathcal{A}$, so $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ is a linear transformation and $\mathcal{B}$ is some basis for $\mathbb{R}^n$. Here we write $\big[T\big]_{\mathcal{B}} = \big[T\big]_{\mathcal{B}\mathcal{B}}$ for the matrix for T relative to $\mathcal{B}$ and $\mathcal{B}$, and call this *the $\mathcal{B}$-matrix for T*.

Consider the $\mathcal{B}$ coord map $\mathbb{R}^n \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^n$. Given $\vec{x}$ and $\vec{y} = T(\vec{x}) = A\vec{x}$, look at

$$\big[\vec{x}\big]_{\mathcal{B}} \text{ and } \big[\vec{y}\big]_{\mathcal{B}} = \big[A\vec{x}\big]_{\mathcal{B}}.$$

Recall that $\vec{x} = P\big[\vec{x}\big]_{\mathcal{B}}$ where $P = P_{\mathcal{E}\mathcal{B}} = \big[\mathcal{B}\big]$. So $\big[\vec{y}\big]_{\mathcal{B}} = P^{-1}\vec{y}$ and we see that

$$\begin{aligned}\big[T\big]_{\mathcal{B}}\big[\vec{x}\big]_{\mathcal{B}} &= \big[T(\vec{x})\big]_{\mathcal{B}} = \big[\vec{y}\big]_{\mathcal{B}} = P^{-1}\vec{y} \\ &= P^{-1}A\vec{x} = P^{-1}AP\big[\vec{x}\big]_{\mathcal{B}}.\end{aligned}$$



Thus the $\mathcal{B}$-matrix for T is $\big[T\big]_{\mathcal{B}} = P^{-1}AP$, so $A = P\big[T\big]_{\mathcal{B}}P^{-1}$. Therefore $A = \big[T\big]_{\mathcal{E}}$ and $\big[T\big]_{\mathcal{B}}$ are *similar* matrices!

All Matrices for a Linear Transformation are Similar

Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be a LT. Suppose $\mathcal{A}, \mathcal{B}$ are any two bases for $\mathbb{R}^n$. Then $\big[T\big]_{\mathcal{A}}$ and $\big[T\big]_{\mathcal{B}}$ are similar matrices.

Above we saw that $\big[T\big]_{\mathcal{E}} = A = P\big[T\big]_{\mathcal{B}}P^{-1}$ where $P = P_{\mathcal{E}\mathcal{B}}$. One way to remember this is by realizing that

$$\big[T\big]_{\mathcal{E}\mathcal{E}} = A = P\big[T\big]_{\mathcal{B}}P^{-1} = P_{\mathcal{E}\mathcal{B}}\big[T\big]_{\mathcal{B}\mathcal{B}}P_{\mathcal{B}\mathcal{E}}.$$

Similarly,

$$\big[T\big]_{\mathcal{A}} = \big[T\big]_{\mathcal{A}\mathcal{A}} = P_{\mathcal{A}\mathcal{B}}\big[T\big]_{\mathcal{B}\mathcal{B}}P_{\mathcal{B}\mathcal{A}} = P_{\mathcal{A}\mathcal{B}}\big[T\big]_{\mathcal{B}}P_{\mathcal{A}\mathcal{B}}^{-1}.$$

It follows that $\big[T\big]_{\mathcal{A}} = P\big[T\big]_{\mathcal{B}}P^{-1}$ where $P = P_{\mathcal{A}\mathcal{B}}$. So, $\big[T\big]_{\mathcal{A}}$ and $\big[T\big]_{\mathcal{B}}$ are similar matrices.

Similar Matrices Represent the Same LT

Suppose the $n \times n$ matrices A and C are similar; so there is an invertible matrix P such that $A = PCP^{-1}$.

Let $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^n$ be given by $T(\vec{x}) = A\vec{x}$.

Since P is invertible, its columns form a basis for $\mathbb{R}^n$. Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_n\}$ where $\vec{b}_j = \mathrm{Col}_j(P)$.

Recall that $P = P_{\mathcal{E}\mathcal{B}}$; so, $P^{-1} = P_{\mathcal{B}\mathcal{E}}$ which means that $\big[\vec{y}\big]_{\mathcal{B}} = P^{-1}\vec{y}$. Thus

$$\begin{aligned}\big[T\big]_{\mathcal{B}}\big[\vec{x}\big]_{\mathcal{B}} = \Big[T(\vec{x})\Big]_{\mathcal{B}} = \Big[A\vec{x}\Big]_{\mathcal{B}} &= P^{-1}\big(A\vec{x}\big)\\ &= P^{-1}\big(PCP^{-1}\vec{x}\big) = C\big(P^{-1}\vec{x}\big) = C\big[\vec{x}\big]_{\mathcal{B}}.\end{aligned}$$

This says that $C = \big[T\big]_{\mathcal{B}}$.

So A and C both represent the LT T.

Conclusion

Two matrices are similar if and only if they represent the same linear transformation.