Similar Matrices and Diagonalization

Linear Algebra
MATH 2076

Similar Matrices and Diagonalizable Matrices

Two $n \times n$ matrices A and B are *similar* if and only if there is an invertible matrix P such that $A = PBP^{-1}$ (and then we also have $B = P^{-1}AP = QAQ^{-1}$ where $Q = P^{-1}$).

An $n \times n$ matrix A is *diagonalizable* if and only if it is similar to a diagonal matrix; that is, there are a diagonal matrix D and an invertible matrix P such that $A = PDP^{-1}$.

An $n \times n$ matrix A is *diagonalizable* if and only if there is an *eigenbasis* assoc'd with A; that is, there is a basis $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ for $\mathbb{R}^n$ such that each vector $\vec{v}_i$ is an eigenvector for A. When this holds, say with $A\vec{v}_i = \lambda_i \vec{v}_i$, we have

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} \vec{v}_1 \ \vec{v}_2 \ \ldots \vec{v}_n \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} \lambda_1 & 0 & \ldots & 0 \\ 0 & \lambda_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & 0 \\ 0 & 0 & \ldots & \lambda_n \end{bmatrix}.$$

So what?

A 2×2 Example

The matrix $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ has *simple* eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$.

Therefore,

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}.$$

Remember what this means: we can easily find the eigencoords for $A\vec{x}$ just by multiplying each eigencoord for $\vec{x}$ by the appropriate eigenvalue. In terms of the eigenbasis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$, $\left[A\vec{x}\right]_{\mathcal{B}} = D\left[\vec{x}\right]_{\mathcal{B}}$.

A 3×3 Example

The matrix $A = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ has *simple* eigenvalues $3, 4, 6$ with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Therefore,

$$A = PDP^{-1} \quad \text{where} \quad P = \begin{bmatrix} 1 & -1 & 1 \\ 1 & 0 & -2 \\ 1 & 1 & 1 \end{bmatrix} \quad \text{and} \quad D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 6 \end{bmatrix}.$$

But what does this mean?

3×3 Matrices with *Simple* and *Double* Eigenvalues

$A_2 = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$ has one *simple* eigenvalue 5 and one *double* eigenvalue 2

with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

$\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an eigenbasis assoc'd with A_2, so A_2 is diagonalizable.

$A_3 = \begin{bmatrix} 5 & -6 & 0 \\ 1 & -2 & 0 \\ 4 & 6 & -1 \end{bmatrix}$ has one *simple* eigenvalue 4 and one *double*

eigenvalue -1 with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

There is **no** eigenbasis assoc'd with A_3, so A_3 is **not** diagonalizable.

Diagonalizable Matrices

An $n \times n$ matrix A is *diagonalizable* if and only if there is an *eigenbasis* assoc'd with A. This holds if, say, A has n distinct (real) eigenvalues, because then the assoc'd eigenvalues are LI and hence form a basis.

In general, there is an *eigenbasis* assoc'd with A if and only if the dimensions of all of the eigenspaces for A add up to n.

Suppose λ is an eigenvalue for A. This means that λ is a zero for the characteristic polynomial $\boldsymbol{p}_A$ of A. Therefore, we can factor $\boldsymbol{p}_A$ as

$$\boldsymbol{p}_A(t) = (t - \lambda)^m \boldsymbol{q}(t) \quad \text{for some } m.$$

We call m the *algebraic multiplicity* of the eigenvalue λ.

We always have $1 \leq \dim \mathbb{E}(\lambda) \leq m$.

We call $\dim \mathbb{E}(\lambda)$ the *geometric multiplicity* of the eigenvalue λ.

There is an *eigenbasis* assoc'd with A if and only if for *every* eigenvalue λ the geometric multiplicity of λ equals the algebraic multiplicity of λ.

Diagonalizable Matrices

An $n \times n$ matrix A is *diagonalizable* if and only if there is an *eigenbasis* assoc'd with A.

There is an *eigenbasis* assoc'd with A if and only if the sum of <u>all</u> the geometric multiplicities equals n.