Using an Eigenbasis to see $\vec{x} \mapsto A\vec{x}$

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. First, $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$. Next,
$\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 8 = (3 - 4\lambda + \lambda^2) - 8 = (\lambda - 5)(\lambda + 1)$.

So, we have *simple* eigenvalues $\lambda = 5$ and $\lambda = -1$.

$\mathbb{E}(5) = \mathcal{NS}(A - 5I) = \mathcal{NS}\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$ and $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$, so
$\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 5$ and $\mathbb{E}(5) = \mathcal{S}pan\{\vec{v}_1\}$.

$\mathbb{E}(-1) = \mathcal{NS}(A + I) = \mathcal{NS}\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
is an eigenvector for $\lambda = -1$ and $\mathbb{E}(-1) = \mathcal{S}pan\{\vec{v}_2\}$.

The eigenspaces $\mathbb{E}(5), \mathbb{E}(-1)$ for A are the lines in $\mathbb{R}^2$ given by

$$\begin{aligned} y &= 2x && \text{for } \mathbb{E}(5) \\ y &= -x && \text{for } \mathbb{E}(-1). \end{aligned}$$

Notice that A's two eigenvectors $\vec{v}_1, \vec{v}_2$ are LI, so form an *eigenbasis*.

Recall that $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ has *simple* eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Since A's two eigenvectors are LI, they form an *eigenbasis* $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$.

Suppose $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$; so $\left[\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Look at

$$A\vec{x} = A\big(c_1\vec{v}_1 + c_2\vec{v}_2\big) = c_1A\vec{v}_1 + c_2A\vec{v}_2 = 5c_1\vec{v}_1 - c_2\vec{v}_2$$

which says that

$$\left[A\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix} 5c_1 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \left[\vec{x}\right]_{\mathcal{B}}.$$

Thus using $\mathcal{B}$-coordinates, the action of A is just multiplication by the diagonal matrix

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

But, how do we get $A\vec{x}$?

From the previous slide: WTF $A\vec{x}$ and we know

$$\left[A\vec{x}\right]_{\mathcal{B}} = D\left[\vec{x}\right]_{\mathcal{B}} \quad \text{where} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

and $\mathcal{B}$ is an eigenbasis assoc'd with A given by

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} \quad \text{where} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \ \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Recall that for any vector $\vec{w}$ in $\mathbb{R}^2$ we have

$$\vec{w} = P\left[\vec{w}\right]_{\mathcal{B}} \quad \text{and} \quad \left[\vec{w}\right]_{\mathcal{B}} = P^{-1}\vec{w}$$

where the $\mathcal{B}$ to $\mathcal{E}$ change of coordinates matrix P is given by

$$P = P_{\mathcal{E}\mathcal{B}} = \left[\vec{v}_1 \ \vec{v}_2\right] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

Thus

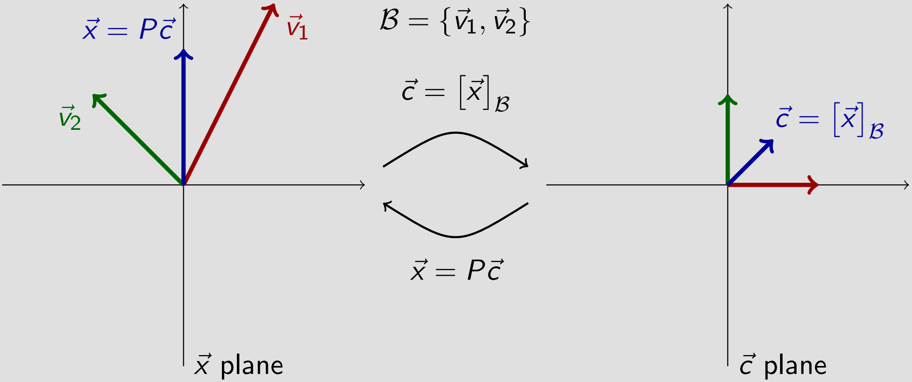
$$\left[\vec{x}\right]_{\mathcal{B}} = P^{-1}\vec{x} \quad \text{and} \quad A\vec{x} = P\left[A\vec{x}\right]_{\mathcal{B}} = PD\left[\vec{x}\right]_{\mathcal{B}} = PDP^{-1}\vec{x}.$$

So,

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \left[\vec{v}_1 \ \vec{v}_2\right], \ D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

So what?

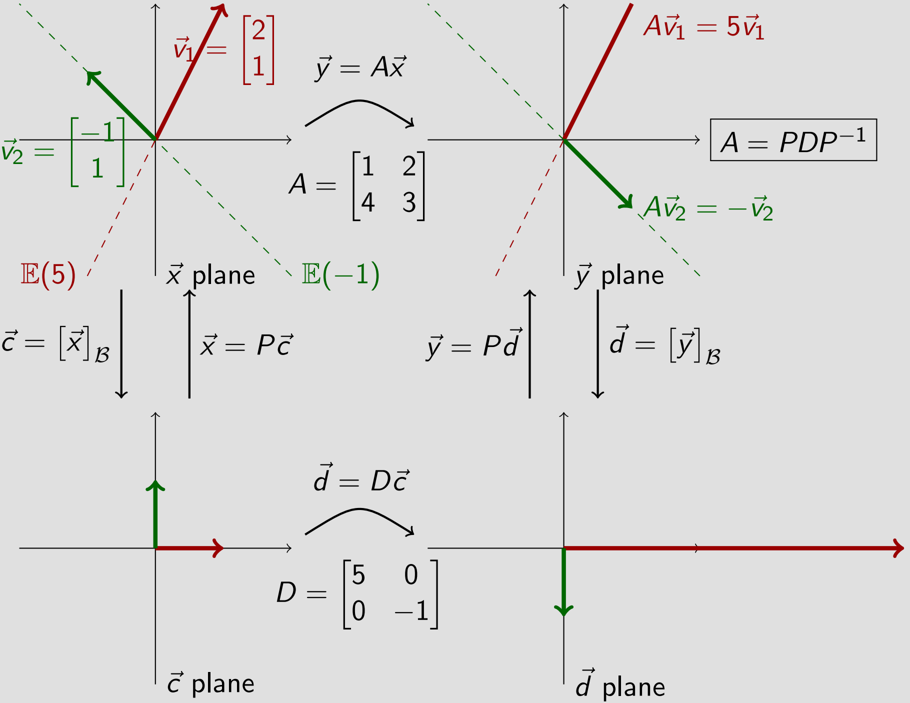
Let's draw pix for the coordinate map $\vec{x} \mapsto [\vec{x}]_{\mathcal{B}}$ and its inverse too.



Here $\vec{c} = [\vec{x}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ means $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2 = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix}\vec{c} = P[\vec{x}]_{\mathcal{B}}$; so the $\mathcal{B}$ to $\mathcal{E}$ change of coordinates matrix P is given by

$$P = P_{\mathcal{E}\mathcal{B}} = \begin{bmatrix} \vec{v}_1 & \vec{v}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

How can we use all this to "see" the LT $\vec{x} \mapsto A\vec{x}$?


$\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$
$\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$
$\mathbb{E}(5)$
$\vec{x}$ plane
$\mathbb{E}(-1)$
$\vec{y} = A\vec{x}$
$A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$
$A\vec{v}_1 = 5\vec{v}_1$
$A = PDP^{-1}$
$A\vec{v}_2 = -\vec{v}_2$
$\vec{y}$ plane
$\vec{c} = [\vec{x}]_{\mathcal{B}}$
$\vec{x} = P\vec{c}$
$\vec{y} = P\vec{d}$
$\vec{d} = [\vec{y}]_{\mathcal{B}}$
$\vec{d} = D\vec{c}$
$D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix}$
$\vec{c}$ plane
$\vec{d}$ plane