Characteristic Equation and Similar Matrices

Linear Algebra
MATH 2076

EigenVectors, EigenValues, EigenSpaces

Definition

Let A be an $n \times n$ matrix. We call $\vec{v}$ an *eigenvector* for A provided

- $\vec{v} \neq \vec{0}$, and
- there is some scalar λ with $A\vec{v} = \lambda\vec{v}$.

When this holds, λ is an *eigenvalue* for A associated to the eigenvector $\vec{v}$ and $\mathbb{E}(\lambda) = \mathcal{NS}(A - \lambda I)$ is the *λ-eigenspace* for A.

Note that

- $\mathbb{E}(\lambda) = \mathcal{NS}(A - \lambda I)$ is always a vector subspace of $\mathbb{R}^n$;
- λ is an eigenvalue for A iff $\det(A - \lambda I) = 0$, and this is the only time $\mathbb{E}(\lambda) \neq \{\vec{0}\}$;
- if λ is an eigenvalue for A, each *non-zero* $\vec{v}$ in $\mathbb{E}(\lambda)$ is an eigenvector for A with assoc'd eigenvalue λ.

Eigen Problems

Given a square matrix A, we want to know how to:

1. Find *all* of the eigenvalues for A.
2. For each eigenvalue for A, find *all* of the assoc'd eigenvectors.
3. Understand the action of A on each of its eigenspaces.

For (1), we just solve $\boxed{\det(A - \lambda I) = 0}$; each solution is an eigenvalue for A. This is the *characteristic equation* of A.

For (2), we just find a basis for $\mathbb{E}(\lambda) = \mathcal{NS}(A - \lambda I)$.

For (3), just note that on $\mathbb{E}(\lambda)$, A acts like the dilation $A\vec{x} = \lambda\vec{x}$ (since each *non-zero* vector in $\mathbb{E}(\lambda)$ is an eigenvector for A).

Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. First, $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$. Next, $\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 8 = (3 - 4\lambda + \lambda^2) - 8 = (\lambda - 5)(\lambda + 1)$.

So, we have *simple* eigenvalues $\lambda = 5$ and $\lambda = -1$.

$\mathbb{E}(5) = \mathcal{NS}(A - 5I) = \mathcal{NS} \begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$ and $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$, so $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector for $\lambda = 5$ and $\mathbb{E}(5) = \mathcal{S}pan\{\vec{v}_1\}$.

$\mathbb{E}(-1) = \mathcal{NS}(A + I) = \mathcal{NS} \begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$ is an eigenvector for $\lambda = -1$ and $\mathbb{E}(-1) = \mathcal{S}pan\{\vec{v}_2\}$.

The eigenspaces $\mathbb{E}(5), \mathbb{E}(-1)$ for A are the lines in $\mathbb{R}^2$ given by

$$y = 2x \qquad \text{for } \mathbb{E}(5)$$
$$y = -x \qquad \text{for } \mathbb{E}(-1).$$

Notice that A's two eigenvectors $\vec{v}_1, \vec{v}_2$ are LI, so form an *eigenbasis*.

Characteristic Polynomials and Equations

Definition

The *characteristic polynomial* for A is $\boxed{\boldsymbol{p}_A(\lambda) = \det(A - \lambda I)}$.

The *characteristic equation* for A is $\boxed{\det(A - \lambda I) = 0}$ (i.e., $\boldsymbol{p}_A(\lambda) = 0$).

For a 2×2 matrix $A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$,

$$\boldsymbol{p}_A(\lambda) = \det(A - \lambda I) = \det \begin{bmatrix} a - \lambda & b \\ c & d - \lambda \end{bmatrix} = (ad - bc) - (a + d)\lambda + \lambda^2.$$

This is a quadratic polynomial, so we can always solve $\boldsymbol{p}_A(\lambda) = 0$, but sometimes the solutions may not be real numbers!

For a 3×3 matrix A, $\boldsymbol{p}_A(\lambda) = \det(A - \lambda I)$ is always a cubic polynomial of the form $\boldsymbol{p}_A(\lambda) = c_0 + c_1\lambda + c_2\lambda^2 - \lambda^3$. Here the characteristic equation $\boldsymbol{p}_A(\lambda) = 0$ **always** has at least one real solution, sometimes two, and sometimes three. So, what does this mean about eigenstuff?

A 3×3 Matrix with Three *Simple* Eigenvalues

$A_1 = \begin{bmatrix} 4 & -1 & 0 \\ -1 & 5 & -1 \\ 0 & -1 & 4 \end{bmatrix}$ has *simple* eigenvalues $3, 4, 6$ with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 1 \\ -2 \\ 1 \end{bmatrix}$.

Here $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an eigenbasis assoc'd with A_1; that is, this is a basis for $\mathbb{R}^3$ consisting of eigenvectors for A.

If $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + c_3\vec{v}_3$, what is $A\vec{v}$? This is easy. We get $A\vec{v} = c_1 A\vec{v}_1 + c_2 A\vec{v}_2 + c_3 A\vec{v}_3 = 3c_1\vec{v}_1 + 4c_2\vec{v}_2 + 6c_3\vec{v}_3$, right?

3×3 Matrices with *Simple* and *Double* Eigenvalues

$A_2 = \begin{bmatrix} 1 & 4 & 4 \\ 0 & 3 & 2 \\ 0 & 2 & 3 \end{bmatrix}$ has one *simple* eigenvalue 5 and one *double* eigenvalue 2

with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 2 \\ 1 \\ 1 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$, $\vec{v}_3 = \begin{bmatrix} 0 \\ 1 \\ -1 \end{bmatrix}$.

Here $\{\vec{v}_1, \vec{v}_2, \vec{v}_3\}$ is an eigenbasis assoc'd with A_2.

$A_3 = \begin{bmatrix} 5 & -6 & 0 \\ 1 & -2 & 0 \\ 4 & 6 & -1 \end{bmatrix}$ has one *simple* eigenvalue 4 and one *double*

eigenvalue -1 with associated eigenvectors $\vec{v}_1 = \begin{bmatrix} 6 \\ 1 \\ 6 \end{bmatrix}$, $\vec{v}_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$.

There is **NO** eigenbasis assoc'd with A_3.

Eigenbases

Let A be an $n \times n$ matrix.

We call $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ an *eigenbasis* assoc'd with A if

- each vector $\vec{v}_i$ is an eigenvector for A, and
- $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_n\}$ is a basis for $\mathbb{R}^n$.

When we have an *eigenbasis* assoc'd with A, it is especially simple to understand the action of the matrix transformation $\vec{x} \mapsto A\vec{x}$.

This is because A is *similar* to a diagonal matrix!

Recall that $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$ has *simple* eigenvalues $\lambda_1 = 5$ and $\lambda_2 = -1$ with assoc'd eigenvectors $\vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ and $\vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Since A's two eigenvectors are LI, they form an *eigenbasis* $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$.

Suppose $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$; so $\left[\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$.

Look at

$$A\vec{x} = A\big(c_1\vec{v}_1 + c_2\vec{v}_2\big) = c_1 A\vec{v}_1 + c_2 A\vec{v}_2 = 5c_1\vec{v}_1 - c_2\vec{v}_2$$

which says that

$$\left[A\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix} 5c_1 \\ -c_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \left[\vec{x}\right]_{\mathcal{B}}.$$

Thus using $\mathcal{B}$-coordinates, the action of A is just multiplication by the diagonal matrix

$$D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

But, how do we get $A\vec{x}$?

From the previous slide: WTF $A\vec{x}$ and we know

$$\left[A\vec{x}\right]_{\mathcal{B}} = D\left[\vec{x}\right]_{\mathcal{B}} \quad \text{where} \quad D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} \quad \text{and} \quad A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$$

and $\mathcal{B}$ is an eigenbasis assoc'd with A given by

$$\mathcal{B} = \{\vec{v}_1, \vec{v}_2\} \quad \text{where} \quad \vec{v}_1 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \quad \vec{v}_2 = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Recall that for any vector $\vec{w}$ in $\mathbb{R}^2$ we have

$$\vec{w} = P\left[\vec{w}\right]_{\mathcal{B}} \quad \text{and} \quad \left[\vec{w}\right]_{\mathcal{B}} = P^{-1}\vec{w}$$

where P is the $\mathcal{B}$ to $\mathcal{E}$ change of coordinates matrix given by

$$P = P_{\mathcal{E}\mathcal{B}} = \left[\vec{v}_1 \ \vec{v}_2\right] = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix}.$$

Thus

$$\left[\vec{x}\right]_{\mathcal{B}} = P^{-1}\vec{x} \quad \text{and} \quad A\vec{x} = P\left[A\vec{x}\right]_{\mathcal{B}} = PD\left[\vec{x}\right]_{\mathcal{B}} = PDP^{-1}\vec{x}.$$

So,

$$A = PDP^{-1} \text{ where } P = \begin{bmatrix} 1 & -1 \\ 2 & 1 \end{bmatrix} = \left[\vec{v}_1 \ \vec{v}_2\right], \ D = \begin{bmatrix} 5 & 0 \\ 0 & -1 \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}.$$

We say that A and D are *similar* matrices.