EigenVectors, EigenValues, EigenSpaces

Linear Algebra
MATH 2076

EigenVectors and EigenValues

Definition

Let A be an $n \times n$ matrix. We call $\vec{v}$ an *eigenvector* for A provided

- $\vec{v} \neq \vec{0}$, and
- there is some scalar λ with $A\vec{v} = \lambda\vec{v}$.

$\vec{0}$ is never an eigenvector

When this holds, λ is an *eigenvalue* for A associated to the eigenvector $\vec{v}$.

Example (A 2×2 matrix with 2 LI eigenvectors)

For $A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$, $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 3 \\ 3 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = \begin{bmatrix} 2 \\ 4 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$.

Therefore, we see that $\vec{v} = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ is an eigenvector with assoc'd eigenvalue 3,

and similarly $\vec{w} = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$ is an eigenvector with assoc'd eigenvalue 2.

See the geogebra file Eigen2x2Ex1gbg.

EigenVectors and EigenValues

Example (Another 2×2 matrix with 2 LI eigenvectors)

The matrix $A = \begin{bmatrix} 9 & 2 \\ -3 & 16 \end{bmatrix}$ has eigenvectors $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix}$.
How do you find the assoc'd eigenvalues? Just multiply, right?

See the geogebra file Eigen3x3Ex1gbg for 3×3 example.

Note that:

1. For an eigenvector $\vec{v}$, we *always* have $\vec{v} \neq \vec{0}$.
2. However, it is possible to have an eigenvalue $\lambda = 0$.
   This happens iff $\mathcal{NS}(A) \neq \{\vec{0}\}$. Right? So?
3. If $\vec{v}$ is an eigenvector for A, then so is $s\vec{v}$ for any scalar $s \neq 0$, and with the same assoc'd eigenvalue.
4. If $\vec{v}, \vec{w}$ are an eigenvectors for A, then so is $\vec{v} + \vec{w}$, and with the same assoc'd eigenvalue.

What do items (3) and (4) above tell us?

How to Find EigenVectors and EigenValues

Let A be an $n \times n$ matrix. Notice that

$$A\vec{v} = \lambda\vec{v} \iff (A - \lambda I)\vec{v} = \vec{0} \iff \vec{v} \text{ is in } \mathcal{NS}(A - \lambda I).$$

Thus,

- $\vec{v}$ is an eigenvector for A iff $\vec{v} \neq \vec{0}$ is in $\mathcal{NS}(A - \lambda I)$, and
- λ is an eigenvalue for A iff $\mathcal{NS}(A - \lambda I) \neq \{\vec{0}\}$.

What does $\mathcal{NS}(M) \neq \{\vec{0}\}$ mean about the matrix M? It says that the columns of M are not LI. For a square matrix M, this is equivalent to saying that M is not invertible. This is equivalent to $\det(M) = 0$.

Thus, λ is an eigenvalue for A iff $\boxed{\det(A - \lambda I) = 0}$. This provides us an *equation* whose solutions are the eigenvalues of A. ☺

Note the role of $\mathcal{NS}(A - \lambda I)$. What can we say about these vectors?

EigenSpaces

Let A be an $n \times n$ matrix. Given any scalar λ, let $\mathbb{E}(\lambda) = \mathcal{NS}(A - \lambda I)$. For most values of λ, $\mathbb{E}(\lambda) = \{\vec{0}\}$. Right? From the previous slide, $\mathbb{E}(\lambda) \neq \{\vec{0}\}$ iff λ is an eigenvalue for A, and then each *non-zero* $\vec{v}$ in $\mathbb{E}(\lambda)$ is an eigenvector for A with assoc'd eigenvalue λ.

Definition

When λ is an eigenvalue for A, we call $\mathbb{E}(\lambda)$ the *λ-eigenspace for A*.

Note that $\mathbb{E}(\lambda) = \mathcal{NS}(A - \lambda I)$ is a vector subspace of $\mathbb{R}^n$. Remember, λ is an eigenvalue for A iff $\det(A - \lambda I) = 0$, and this is the only time $\mathbb{E}(\lambda) \neq \{\vec{0}\}$.

Example (A 2×2 matrix with 2 LI eigenvectors)

$A = \begin{bmatrix} 4 & -1 \\ 2 & 1 \end{bmatrix}$ has $A \begin{bmatrix} 1 \\ 1 \end{bmatrix} = 3 \begin{bmatrix} 1 \\ 1 \end{bmatrix}$ and $A \begin{bmatrix} 1 \\ 2 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 2 \end{bmatrix}$; i.e., EVs 2 and 3.

So, $\mathbb{E}(3) = \mathcal{S}pan\left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix} \right\}$ and $\mathbb{E}(2) = \mathcal{S}pan\left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix} \right\}$. These are lines in $\mathbb{R}^2$.

$\lambda = 2$ is an eigenvalue for $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$. Find all assoc'd eigenvectors.

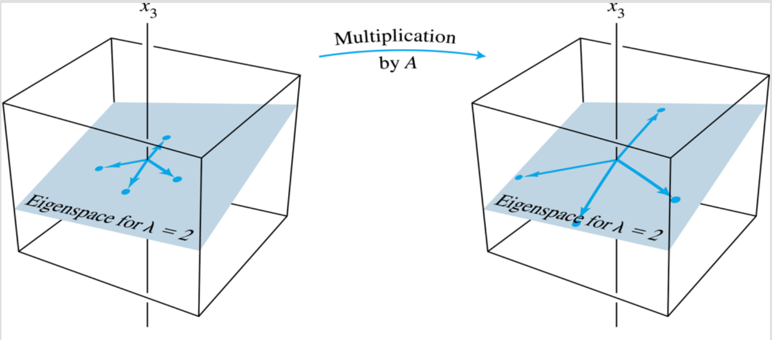
We seek a basis for the eigenspace $\mathbb{E}(2) = \mathcal{NS}(A - 2I)$. Look at $A - 2I = \begin{bmatrix} 2 & -1 & 6 \\ 2 & -1 & 6 \\ 2 & -1 & 6 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 & 6 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$. Evidently, x_2 and x_3 are free; say $x_2 = 2s$ and $x_3 = t$. Then $2x_1 - 2s + 6t = 0$, and the general solution to $(A - 2I)\vec{x} = \vec{0}$ has the form $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = \begin{bmatrix} s - 3t \\ 2s \\ t \end{bmatrix} = s \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix} + t \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

So $\mathcal{B} = \left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{E}(2)$ and we see that $\mathbb{E}(2)$ is the plane in $\mathbb{R}^3$ spanned by the two LI eigenvectors $\begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix}$.

Note that on $\mathbb{E}(2)$, A acts like the dilation $\vec{x} \mapsto A\vec{x} = 2\vec{x}$.

Action of A on its Eigenspace

The matrix $A = \begin{bmatrix} 4 & -1 & 6 \\ 2 & 1 & 6 \\ 2 & -1 & 8 \end{bmatrix}$ has an eigenvalue $\lambda = 2$ and $\mathbb{E}(2)$ is the plane in $\mathbb{R}^3$ given by $\mathbb{E}(2) = Span\left\{ \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} -3 \\ 0 \\ 1 \end{bmatrix} \right\}$.



A acts as a dilation on the eigenspace.

See Eigen3x3Ex1.ggb

Eigen Problems

Given a square matrix A, we want to know how to:

1. Find *all* of the eigenvalues for A.
2. For each eigenvalue for A, find *all* of the assoc'd eigenvectors.
3. Understand the action of A on each of its eigenspaces.

For item (1), we just solve $\boxed{\det(A - \lambda I) = 0}$; each solution is an eigenvalue for A.

For item (2), we just find a basis for each $\mathbb{E}(\lambda) = \mathcal{NS}(A - \lambda I)$.

For item (3), just note that on $\mathbb{E}(\lambda)$, A acts like the dilation $A\vec{x} = \lambda\vec{x}$ (since each *non-zero* vector in $\mathbb{E}(\lambda)$ is an eigenvector for A).

Let $A = \begin{bmatrix} 1 & 2 \\ 4 & 3 \end{bmatrix}$. First, $A - \lambda I = \begin{bmatrix} 1-\lambda & 2 \\ 4 & 3-\lambda \end{bmatrix}$. Next,
$\det(A - \lambda I) = (1-\lambda)(3-\lambda) - 8 = (3 - 4\lambda + \lambda^2) - 8 = (\lambda - 5)(\lambda + 1)$.

So, we have *simple* eigenvalues $\lambda = 5$ and $\lambda = -1$.

$\mathbb{E}(5) = \mathcal{NS}(A - 5I) = \mathcal{NS}\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix}$ and $\begin{bmatrix} -4 & 2 \\ 4 & -2 \end{bmatrix} \sim \begin{bmatrix} 2 & -1 \\ 0 & 0 \end{bmatrix}$, so
$\mathbb{E}(5) = \mathcal{S}pan\{\begin{bmatrix} 1 \\ 2 \end{bmatrix}\}$. Check this!

$\mathbb{E}(-1) = \mathcal{NS}(A + I) = \mathcal{NS}\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix}$ and $\begin{bmatrix} 2 & 2 \\ 4 & 4 \end{bmatrix} \sim \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$, so
$\mathbb{E}(-1) = \mathcal{S}pan\{\begin{bmatrix} 1 \\ -1 \end{bmatrix}\}$. Check this!

Thus A has two eigenspaces which are the lines in $\mathbb{R}^2$ given by

$$y = 2x \qquad \text{for } \mathbb{E}(5)$$
$$y = -x \qquad \text{for } \mathbb{E}(-1).$$

Notice that the two eigenvectors for A are LI, so form a basis for $\mathbb{R}^2$.