Change of Basis and Coordinates

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Coordinates and Coordinate Vectors

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We call $c_1, c_2, \dots, c_k$ the *$\mathcal{B}$-coordinates of $\vec{v}$* and $\left[\vec{v}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ is the *$\mathcal{B}$-coordinate vector for $\vec{v}$*.

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Note that $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^k$.

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Again, while $\vec{v}$ is a vector in $\mathbb{R}^n$, $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^k$.

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The inverse of the $\mathcal{B}$-coordinate mapping is the linear transformation $\mathbb{R}^k \xrightarrow{T} \mathbb{V}$ given by the formula

$$T(\vec{x}) = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_k\vec{b}_k \quad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}.$$

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This is why the $\mathcal{B}$-coord mapping $\vec{v} \mapsto \left[\vec{v}\right]_{\mathcal{B}}$ is a linear transformation.

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How are the coordinate maps $\vec{v} \mapsto [\vec{v}]_{\mathcal{B}}$ and $\vec{v} \mapsto [\vec{v}]_{\mathcal{A}}$ related?

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Remember, $\vec{x} = \left[\vec{b}_1 \ \vec{b}_2\right] \left[\vec{x}\right]_{\mathcal{B}}$, right? Letting $P = \left[\vec{b}_1 \ \vec{b}_2\right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ we have $\vec{x} = P\left[\vec{x}\right]_{\mathcal{B}}$. We call P the *$\mathcal{B}$ to $\mathcal{E}$ change of coordinates matrix*, and write $P = P_{\mathcal{E}\mathcal{B}}$. Thus $\boxed{\vec{x} = P_{\mathcal{E}\mathcal{B}}\left[\vec{x}\right]_{\mathcal{B}}}$.

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Notice that P is invertible. Therefore, $P^{-1}\vec{x} = \left[\vec{x}\right]_{\mathcal{B}}$.

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Notice that P is invertible. Therefore, $P^{-1}\vec{x} = \left[\vec{x}\right]_{\mathcal{B}}$. Thus P^{-1} is the *$\mathcal{E}$ to $\mathcal{B}$ change of coordinates matrix*, that is, $\boxed{P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1}}$.

The $\mathcal{B}$ to $\mathcal{A}$ Change of Coordinates Matrix

Let $\mathcal{A}$ and $\mathcal{B}$ be bases for a vector space $\mathbb{V}$. Suppose $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_k\}$.

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$$\left[\vec{v}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

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So, $\vec{v} = \sum_{i=1}^{k} c_i \vec{b}_i$.

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So, $\vec{v} = \sum_{i=1}^{k} c_i \vec{b}_i$.

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It follows that

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$$[\vec{v}]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

So, $\vec{v} = \sum_{i=1}^{k} c_i \vec{b}_i$.

It follows that

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$$= \left[\left[\vec{b}_1\right]_{\mathcal{A}} \cdots \left[\vec{b}_k\right]_{\mathcal{A}} \right] \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix}$$

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Writing $P = P_{\mathcal{A}\mathcal{B}}$

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Writing $P = P_{\mathcal{AB}}$ get $\boxed{\left[\vec{v}\right]_{\mathcal{A}} = P_{\mathcal{AB}}\left[\vec{v}\right]_{\mathcal{B}}}$ and $\boxed{P_{\mathcal{AB}} = \left[\left[\vec{b}_1\right]_{\mathcal{A}} \ldots \left[\vec{b}_k\right]_{\mathcal{A}}\right]}$.

Picturing Change of Coordinates

Let $\mathcal{A}$ and $\mathcal{B}$ be bases for a vector space $\mathbb{V}$.

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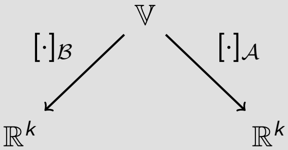
Let $\mathcal{A}$ and $\mathcal{B}$ be bases for a vector space $\mathbb{V}$. Consider the $\mathcal{B}$ and $\mathcal{A}$ coordinate mappings from $\mathbb{V}$ to $\mathbb{R}^k$.



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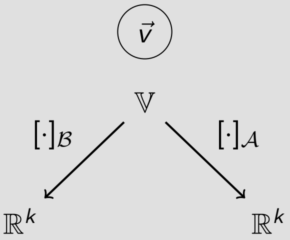

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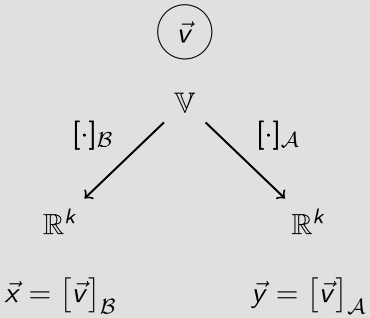
$$\vec{x} = \left[\vec{v}\right]_{\mathcal{B}} \text{ and } \vec{y} = \left[\vec{v}\right]_{\mathcal{A}}.$$



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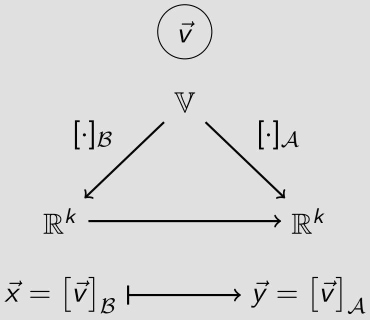
Consider the LT $\mathbb{R}^k \to \mathbb{R}^k$ given by $\vec{x} \mapsto \vec{y}$.



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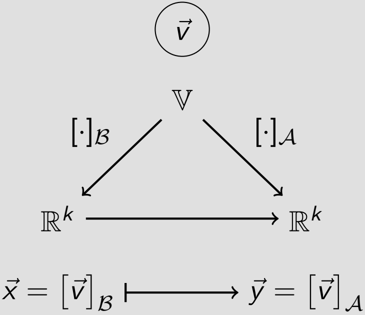
This is a matrix transformation, and evidently



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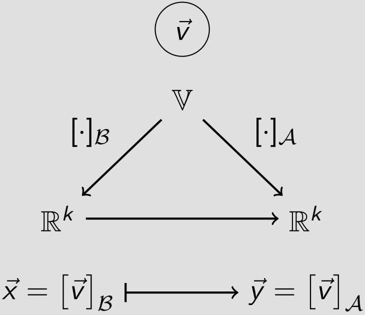

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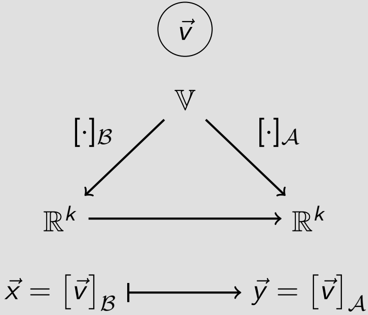

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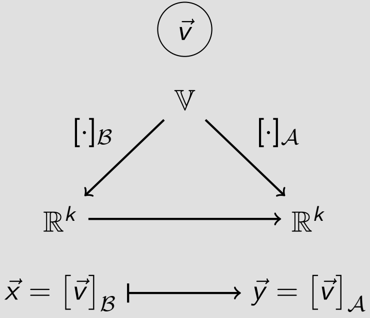
This is a matrix transformation, and evidently $\vec{y} = \left[\vec{v}\right]_{\mathcal{A}} = P\left[\vec{v}\right]_{\mathcal{A}} = P\vec{x}$ where $P = P_{\mathcal{AB}}$.



Example

Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}}$ where $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

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$$P = \left[\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3\right] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = P_{\mathcal{E}\mathcal{B}},$$

Example

Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}}$ where $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

$P = \left[\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3\right] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = P_{\mathcal{E}\mathcal{B}}$, so $P_{\mathcal{B}\mathcal{E}} = P^{-1} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

Example

Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}}$ where $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

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Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x - 2y + z \\ -2x + 2y - z \\ x - y + z \end{bmatrix}$$

Example

Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}}$ where $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

$P = [\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3] = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = P_{\mathcal{E}\mathcal{B}}$, so $P_{\mathcal{B}\mathcal{E}} = P^{-1} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x - 2y + z \\ -2x + 2y - z \\ x - y + z \end{bmatrix}$$

For example, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.