Change of Basis and Coordinates

Linear Algebra
MATH 2076

Coordinates and Coordinate Vectors

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_k\}$ be a basis for a vector space $\mathbb{V}$. Then for each vector $\vec{v}$ in $\mathbb{V}$, there are *unique* scalars $c_1, c_2, \ldots, c_k$ such that

$$\vec{v} = c_1\vec{b}_1 + c_2\vec{b}_2 + \cdots + c_k\vec{b}_k = \sum_{i=1}^{k} c_i\vec{b}_i.$$

We call $c_1, c_2, \ldots, c_k$ the *$\mathcal{B}$-coordinates of $\vec{v}$* and $\left[\vec{v}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$ is the *$\mathcal{B}$-coordinate vector for $\vec{v}$*.

Note that $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^k$.

Coordinates for Subspaces of $\mathbb{R}^n$

Suppose $\mathbb{V}$ is a vector subspace of $\mathbb{R}^n$.

In this setting, finding coord vectors $\left[\vec{v}\right]_{\mathcal{B}}$ (for $\vec{v}$ in $\mathbb{V}$) is just the problem of solving $B\vec{x} = \vec{v}$ where $B = \left[\vec{b}_1\ \vec{b}_2 \cdots \vec{b}_k\right]$.

Given a vector $\vec{v}$ in $\mathbb{V}$, $\left[\vec{v}\right]_{\mathcal{B}}$ is the unique solution to $B\vec{x} = \vec{v}$. This holds because if $\vec{v} = B\vec{x} = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_k\vec{b}_k$, then $x_1, x_2, \ldots, x_k$ are the $\mathcal{B}$-coords of $\vec{v}$.

Again, while $\vec{v}$ is a vector in $\mathbb{R}^n$, $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^k$.

Coordinate Mappings

Let $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_k\}$ be a basis for a vector space $\mathbb{V}$. Then each $\vec{v}$ in $\mathbb{V}$ has an associated *$\mathcal{B}$-coordinate vector* $\left[\vec{v}\right]_{\mathcal{B}}$ where $c_1, c_2, \ldots, c_k$ are the *$\mathcal{B}$-coordinates of* $\vec{v}$. Again, $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^k$.

$$\left[\vec{v}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

The *$\mathcal{B}$-coordinate mapping* is the LT $\mathbb{V} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^k$ given by the formula $\vec{v} \mapsto \left[\vec{v}\right]_{\mathcal{B}}$.

The inverse of the $\mathcal{B}$-coordinate mapping is the linear transformation $\mathbb{R}^k \xrightarrow{T} \mathbb{V}$ given by the formula

$$T(\vec{x}) = x_1\vec{b}_1 + x_2\vec{b}_2 + \cdots + x_k\vec{b}_k \quad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_k \end{bmatrix}.$$

Thus $\vec{x}$ in $\mathbb{R}^k$ is mapped to $\vec{v} = T(\vec{x})$ in $\mathbb{V}$ and $\left[\vec{v}\right]_{\mathcal{B}} = \vec{x}$. That is, $\left[T(\vec{x})\right]_{\mathcal{B}} = \vec{x}$.

Two Bases

Suppose we have two bases, say $\mathcal{A}$ and $\mathcal{B}$ for a vector space $\mathbb{V}$.

How are the coordinate vectors $[\vec{v}]_{\mathcal{B}}$ and $[\vec{v}]_{\mathcal{A}}$ related to each other?

If we know one coordinate vector, how do we get the other?

How are the coordinate maps $\vec{v} \mapsto [\vec{v}]_{\mathcal{B}}$ and $\vec{v} \mapsto [\vec{v}]_{\mathcal{A}}$ related?

Example

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^2$ and $\begin{bmatrix} 1 \\ 3 \end{bmatrix} = 2 \begin{bmatrix} 1 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 1 \end{bmatrix}$, so $\begin{bmatrix} 1 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ 1 \end{bmatrix}$. What are the $\mathcal{B}$-coord vectors for $\begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 10 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}$?

We could just solve the appropriate SLEs, but is there a better way?

Remember, $\vec{x} = \left[\vec{b}_1 \ \vec{b}_2\right] \left[\vec{x}\right]_{\mathcal{B}}$, right? Letting $P = \left[\vec{b}_1 \ \vec{b}_2\right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ we have $\vec{x} = P\left[\vec{x}\right]_{\mathcal{B}}$. We call P the *$\mathcal{B}$ to $\mathcal{E}$ change of coordinates matrix*, and write $P = P_{\mathcal{E}\mathcal{B}}$. Thus $\boxed{\vec{x} = P_{\mathcal{E}\mathcal{B}}\left[\vec{x}\right]_{\mathcal{B}}}$.

Notice that P is invertible. Therefore, $P^{-1}\vec{x} = \left[\vec{x}\right]_{\mathcal{B}}$. Thus P^{-1} is the *$\mathcal{E}$ to $\mathcal{B}$ change of coordinates matrix*, that is, $\boxed{P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1}}$.

Example (continued)

For $\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}$, $P = P_{\mathcal{E}\mathcal{B}} = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 \end{bmatrix} = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is the *$\mathcal{B}$ to $\mathcal{E}$ change of coordinates matrix* and $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = P^{-1}$ is the *$\mathcal{E}$ to $\mathcal{B}$ change of coordinates matrix*.

We find that $P^{-1} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$, so

$$\begin{bmatrix} x \\ y \end{bmatrix}_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}} \begin{bmatrix} x \\ y \end{bmatrix} = P^{-1} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \frac{1}{2} \begin{bmatrix} x + y \\ y - x \end{bmatrix}.$$

$\mathcal{A} = \{\vec{a}_1, \vec{a}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ is also a basis for $\mathbb{R}^2$.

What are the $\mathcal{A}$-coord vectors for $\begin{bmatrix} -4 \\ 2 \end{bmatrix}, \begin{bmatrix} -7 \\ 10 \end{bmatrix}, \begin{bmatrix} x \\ y \end{bmatrix}$?

How do $\mathcal{A}$-coords and $\mathcal{B}$-coords compare?

Example (continued)

$\mathcal{B} = \{\vec{b}_1, \vec{b}_2\} = \left\{ \begin{bmatrix} 1 \\ 1 \end{bmatrix}, \begin{bmatrix} -1 \\ 1 \end{bmatrix} \right\}, \mathcal{A} = \{\vec{a}_1, \vec{a}_2\} = \left\{ \begin{bmatrix} 1 \\ 2 \end{bmatrix}, \begin{bmatrix} -3 \\ 4 \end{bmatrix} \right\}$ bases of $\mathbb{R}^2$

$P = P_{\mathcal{E}\mathcal{B}} = \left[\vec{b}_1 \ \vec{b}_2 \right] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ is the $\mathcal{B}$ *to* $\mathcal{E}$ *change of coordinates matrix*

and $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = P^{-1}$ is the $\mathcal{E}$ *to* $\mathcal{B}$ *change of coordinates matrix.*

$Q = P_{\mathcal{E}\mathcal{A}} = \left[\vec{a}_1 \ \vec{a}_2 \right] = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ is the $\mathcal{A}$ *to* $\mathcal{E}$ *change of coords matrix* and

$P_{\mathcal{A}\mathcal{E}} = P_{\mathcal{E}\mathcal{A}}^{-1} = Q^{-1} = \frac{1}{10} \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ is the $\mathcal{E}$ *to* $\mathcal{A}$ *change of coords matrix.*

We compute $\begin{bmatrix} -4 \\ 2 \end{bmatrix}_{\mathcal{A}}$ in two different ways. First (as above):

$$\begin{bmatrix} -4 \\ 2 \end{bmatrix}_{\mathcal{A}} = P_{\mathcal{A}\mathcal{E}} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = Q^{-1} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \frac{1}{10} \begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix} \begin{bmatrix} -4 \\ 2 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

Example—second way to compute $\begin{bmatrix} -4 \\ 2 \end{bmatrix}_{\mathcal{A}}$

$P = P_{\mathcal{E}\mathcal{B}} = [\vec{b}_1 \ \vec{b}_2] = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$ and $P_{\mathcal{B}\mathcal{E}} = P_{\mathcal{E}\mathcal{B}}^{-1} = P^{-1} = \frac{1}{2}\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}$ are the $\mathcal{B}$ to $\mathcal{E}$ and $\mathcal{E}$ to $\mathcal{B}$ change of coords matrices.
$Q = P_{\mathcal{E}\mathcal{A}} = [\vec{a}_1 \ \vec{a}_2] = \begin{bmatrix} 1 & -3 \\ 2 & 4 \end{bmatrix}$ and $P_{\mathcal{A}\mathcal{E}} = P_{\mathcal{E}\mathcal{A}}^{-1} = Q^{-1} = \frac{1}{10}\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}$ are the $\mathcal{A}$ to $\mathcal{E}$ and $\mathcal{E}$ to $\mathcal{A}$ change of coords matrices.

We find that $\begin{bmatrix} -4 \\ 2 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$. Next, the *$\mathcal{B}$ to $\mathcal{A}$ change of coords matrix* is $P_{\mathcal{A}\mathcal{B}} = P_{\mathcal{A}\mathcal{E}}P_{\mathcal{E}\mathcal{B}} = Q^{-1}P = \frac{1}{10}\begin{bmatrix} 4 & 3 \\ -2 & 1 \end{bmatrix}\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} 7 & -1 \\ -1 & 3 \end{bmatrix}$.
Thus:

$$\begin{bmatrix} -4 \\ 2 \end{bmatrix}_{\mathcal{A}} = P_{\mathcal{A}\mathcal{B}}\begin{bmatrix} -4 \\ 2 \end{bmatrix}_{\mathcal{B}} = \frac{1}{10}\begin{bmatrix} 7 & -1 \\ -1 & 3 \end{bmatrix}\begin{bmatrix} -1 \\ 3 \end{bmatrix} = \frac{1}{10}\begin{bmatrix} -10 \\ 10 \end{bmatrix} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}.$$

The $\mathcal{B}$ to $\mathcal{A}$ Change of Coordinates Matrix

Let $\mathcal{A}$ and $\mathcal{B}$ be bases for a vector space $\mathbb{V}$. Suppose $\mathcal{B} = \{\vec{b}_1, \ldots, \vec{b}_k\}$.
Then each $\vec{v}$ in $\mathbb{V}$ has an associated *$\mathcal{B}$-coordinate vector* $\left[\vec{v}\right]_{\mathcal{B}}$

$$\left[\vec{v}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_k \end{bmatrix}$$

where $c_1, c_2, \ldots, c_k$ are the *$\mathcal{B}$-coordinates of $\vec{v}$*.
So, $\vec{v} = \sum_{i=1}^{k} c_i \vec{b}_i$.
It follows that

$$\begin{aligned} \left[\vec{v}\right]_{\mathcal{A}} &= \left[\sum_{i=1}^{k} c_i \vec{b}_i\right]_{\mathcal{A}} = \sum_{i=1}^{k} c_i \left[\vec{b}_i\right]_{\mathcal{A}} \\ &= \left[\left[\vec{b}_1\right]_{\mathcal{A}} \ldots \left[\vec{b}_k\right]_{\mathcal{A}}\right] \begin{bmatrix} c_1 \\ \vdots \\ c_k \end{bmatrix} \\ &= P\left[\vec{v}\right]_{\mathcal{B}}; \end{aligned}$$

here $P = \left[\left[\vec{b}_1\right]_{\mathcal{A}} \ldots \left[\vec{b}_k\right]_{\mathcal{A}}\right]$ is the *$\mathcal{B}$ to $\mathcal{A}$ change of coordinates matrix*.

Writing $P = P_{\mathcal{AB}}$ get $\boxed{\left[\vec{v}\right]_{\mathcal{A}} = P_{\mathcal{AB}}\left[\vec{v}\right]_{\mathcal{B}}}$ and $\boxed{P_{\mathcal{AB}} = \left[\left[\vec{b}_1\right]_{\mathcal{A}} \ldots \left[\vec{b}_k\right]_{\mathcal{A}}\right]}$.

Picturing Change of Coordinates

Let $\mathcal{A}$ and $\mathcal{B}$ be bases for a vector space $\mathbb{V}$. Consider the $\mathcal{B}$ and $\mathcal{A}$ coordinate mappings from $\mathbb{V}$ to $\mathbb{R}^k$. Given $\vec{v}$ in $\mathbb{V}$, let

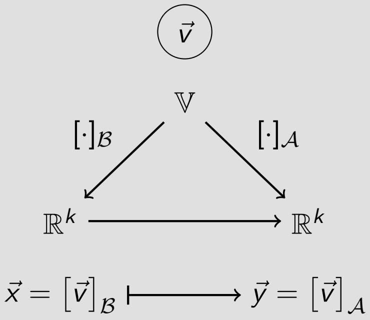
$$\vec{x} = \left[\vec{v}\right]_{\mathcal{B}} \text{ and } \vec{y} = \left[\vec{v}\right]_{\mathcal{A}}.$$

Consider the LT $\mathbb{R}^k \to \mathbb{R}^k$ given by $\vec{x} \mapsto \vec{y}$.

This is the linear transformation given by

$$\vec{y} = \left[\vec{v}\right]_{\mathcal{A}} = P\left[\vec{v}\right]_{\mathcal{B}} = P\vec{x}$$

where $P = P_{\mathcal{A}\mathcal{B}}$.



Example

Find $\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}}$ where $\mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \\ 2 \end{bmatrix} \right\}$ is a basis for $\mathbb{R}^3$.

$P = \begin{bmatrix} \vec{b}_1 & \vec{b}_2 & \vec{b}_3 \end{bmatrix} = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 2 & 1 \\ 0 & 1 & 2 \end{bmatrix} = P_{\mathcal{E}\mathcal{B}}$, so $P_{\mathcal{B}\mathcal{E}} = P^{-1} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix}$.

Thus

$$\begin{bmatrix} x \\ y \\ z \end{bmatrix}_{\mathcal{B}} = P_{\mathcal{B}\mathcal{E}} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3 & -2 & 1 \\ -2 & 2 & -1 \\ 1 & -1 & 1 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 3x - 2y + z \\ -2x + 2y - z \\ x - y + z \end{bmatrix}$$

For example, $\begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}_{\mathcal{B}} = \begin{bmatrix} 2 \\ -1 \\ 2 \end{bmatrix}$.

Example

$$\mathcal{A} = \{\vec{a}_1, \vec{a}_2, \vec{a}_3\} = \left\{ \begin{bmatrix}1\\1\\1\end{bmatrix}, \begin{bmatrix}0\\1\\1\end{bmatrix}, \begin{bmatrix}0\\0\\1\end{bmatrix} \right\}, \mathcal{B} = \{\vec{b}_1, \vec{b}_2, \vec{b}_3\} = \left\{ \begin{bmatrix}1\\1\\0\end{bmatrix}, \begin{bmatrix}1\\2\\1\end{bmatrix}, \begin{bmatrix}0\\1\\2\end{bmatrix} \right\}$$

are bases for $\mathbb{R}^3$. Find $P_{\mathcal{AB}}$.

Since $P_{\mathcal{AB}} = \left[[\vec{b}_1]_{\mathcal{A}} [\vec{b}_2]_{\mathcal{A}} [\vec{b}_3]_{\mathcal{A}} \right]$, we could just find the $\mathcal{A}$-coord vectors for each of $\vec{b}_1, \vec{b}_2, \vec{b}_3$; this is one way to proceed.

Alternatively, $P_{\mathcal{AB}} = P_{\mathcal{AE}} P_{\mathcal{EB}}$, where $P_{\mathcal{EB}} = \left[\vec{b}_1 \ \vec{b}_2 \ \vec{b}_3\right] = \begin{bmatrix}1 & 1 & 0\\1 & 2 & 1\\0 & 1 & 2\end{bmatrix}$, and

$$P_{\mathcal{AE}} = P_{\mathcal{EA}}^{-1} = \left[\vec{a}_1 \ \vec{a}_2 \ \vec{a}_3\right]^{-1} = \begin{bmatrix}1 & 0 & 0\\1 & 1 & 0\\1 & 1 & 1\end{bmatrix}^{-1} = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix}.$$

Thus

$$P_{\mathcal{AB}} = \begin{bmatrix}1 & 0 & 0\\-1 & 1 & 0\\0 & -1 & 1\end{bmatrix} \begin{bmatrix}1 & 1 & 0\\1 & 2 & 1\\0 & 1 & 2\end{bmatrix} = \begin{bmatrix}1 & 1 & 0\\0 & 1 & 1\\-1 & -1 & 1\end{bmatrix}.$$