Dimension, Rank, Nullity, and the Rank-Nullity Theorem

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Basic Facts About Bases

Let $\mathbb{V}$ be a non-trivial vector space; so $\mathbb{V} \neq \{\vec{0}\}$. Then:

- $\mathbb{V}$ has a basis, and,
- any two bases for $\mathbb{V}$ contain the same number of vectors.

Definition

If $\mathbb{V}$ has a finite basis, we call $\mathbb{V}$ *finite dimensional*; otherwise, we say that $\mathbb{V}$ is *infinite dimensional*.

Definition

If $\mathbb{V}$ is *finite dimensional*, then the *dimension of* $\mathbb{V}$ is the number of vectors in any basis for $\mathbb{V}$; we write $\dim \mathbb{V}$ for the dimension of $\mathbb{V}$.

The *dimension* of the trivial vector space $\{\vec{0}\}$ is defined to be 0.

Dimension Examples

Examples

- $\mathbb{R}^n$ has dimension n, bcuz $\mathcal{S} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ is a basis for $\mathbb{R}^n$
- $\mathbb{P}_n$ has dimension $n+1$, bcuz $\mathcal{P} = \{\mathbf{1}, \boldsymbol{t}, \boldsymbol{t}^2, \ldots, \boldsymbol{t}^n\}$ is a basis for $\mathbb{P}_n$
- $\mathbb{R}^\infty$ is infinite dimensional
- $\mathbb{P}$ is infinite dimensional
- If $\{\vec{a}_1, \ldots, \vec{a}_p\}$ is a LI set of vectors in $\mathbb{R}^n$, then $\mathbb{V} = \mathcal{S}pan\{\vec{a}_1, \ldots, \vec{a}_p\}$ is a p-dimensional vector subspace of $\mathbb{R}^n$. We call $\mathbb{V}$ a *p-plane in $\mathbb{R}^n$*.

Examples

Let $\mathbb{U}^{2\times 2}$ and $\mathbb{S}^{2\times 2}$ be the spaces of all upper triangular and all symmetric 2×2 matrices, respectively. Let's find dim $\mathbb{U}^{2\times 2}$ and dim $\mathbb{S}^{2\times 2}$. We just need bases, right?

Next, what does a symmetric 2×2 matrix look like? Just $\begin{bmatrix} a & b \\ b & c \end{bmatrix}$, right? But

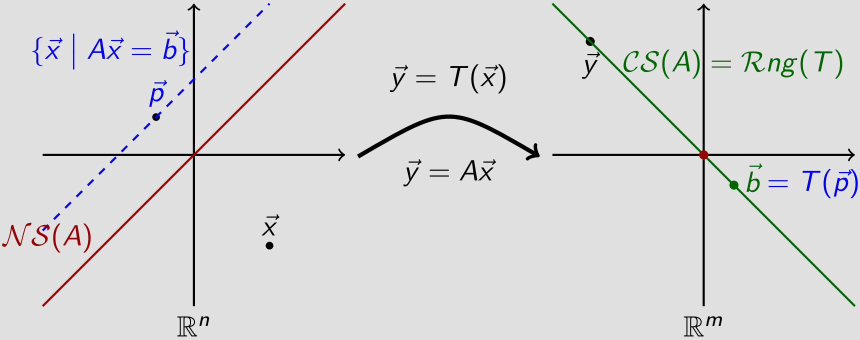
$$\begin{bmatrix} a & b \\ b & c \end{bmatrix} = a \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} + b \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + c \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

so the three matrices on the above right certainly span $\mathbb{S}^{2\times 2}$. It's not hard to see that they are LI, so they form a basis. Therefore, dim $\mathbb{S}^{2\times 2} = 3$.

What about upper triangular and symmetric $n \times n$ matrices?

$A = [\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n]$ an $m \times n$ matrix and $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

$$\begin{aligned}\mathcal{NS}(A) &= \{\vec{x} \mid A\vec{x} = \vec{0}\} \quad \text{and} \\ \mathcal{CS}(A) &= \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\} \\ &= \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\} \\ &= \mathcal{R}ng(T)\end{aligned}$$



Dimensions of Null Space and Column Space

Gotta find bases for the null space $\mathcal{NS}(A)$ and column space $\mathcal{CS}(A)$ of A. Just:

- row reduce A to E, a REF (or RREF) for A
- columns of E with row leaders correspond to *pivot* columns of A
- the *pivot* columns of A are LI and span $\mathcal{CS}(A)$, so form a basis
- write the SS for $A\vec{x} = \vec{0}$ in parametric vector form
- identify LI vectors that span $\mathcal{NS}(A)$, these form a basis

So,

$$\begin{aligned} \dim \mathcal{CS}(A) &= \#\text{ of pivot cols of } A \\ &= \#\text{ of row leaders in } E \\ &= \#\text{ of non-zero rows in } E \\ &= r \end{aligned}$$

and

$$\begin{aligned} \dim \mathcal{NS}(A) &= \#\text{ of free variables} \\ &= \#\text{ of cols of } A - r \\ &= n - r = q. \end{aligned}$$

Notice that $r + q = n = \#$ of columns of A.

Example—Null Space and Column Space

Find the dimensions of the null space and column space of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

Using elem row ops, we find the indicated REF and RREF for A.

Thus columns 1,2,4 are pivot columns for A, so dim $\mathcal{CS}(A) = 3$.

There are two free variables (x_3 and x_5), so dim $\mathcal{NS}(A) = 2$.

Notice that $3 + 2 = 5 = \#$ of columns of A.

Rank, Nullity, and the Rank-Nullity Theorem

Let A be an $m \times n$ matrix.

The dimension of $\mathcal{CS}(A)$ is called the *rank* of A; $\text{rank}(A) = \dim \mathcal{CS}(A)$.

The dimension of $\mathcal{NS}(A)$ is called the *nullity* of A; $\text{null}(A) = \dim \mathcal{NS}(A)$.
So,

$$\begin{aligned} r &= \text{rank}(A) = \dim \mathcal{CS}(A) = \# \text{ of pivot columns of } A, \\ q &= \text{null}(A) = \dim \mathcal{NS}(A) = \# \text{ of free variables} \end{aligned}$$

and

$$\text{rank}(A) + \text{null}(A) = r + q = n = \# \text{ of columns of } A.$$

This last fact is called the *Rank-Nullity Theorem*.

Having the Right Number of Vectors

Let $\mathbb{V}$ be a vector space. Recall that $\mathcal{B}$ is a basis for $\mathbb{V}$ iff both $\mathcal{B}$ is LI and $\mathbb{V} = \mathcal{S}pan\,\mathcal{B}$.

Suppose we know that dim $\mathbb{V} = p$. Let $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ be any vectors in $\mathbb{V}$. The following are equivalent:

- $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is a basis for $\mathbb{V}$
- $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is LI
- $\mathbb{V} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$

If we know the dimension ahead of time, it is easier to find a basis.

The Rank-Nullity Theorem helps here!

Example

Suppose A is a 20×17 matrix. What can we say about $A\vec{x} = \vec{b}$?

Recall that $\mathcal{NS}(A)$ is a subspace of $\mathbb{R}^{17}$ and $\mathcal{CS}(A)$ is a subspace of $\mathbb{R}^{20}$.

Since $\text{rank}(A) + \text{null}(A) = 17$, $\dim \mathcal{CS}(A) = \text{rank}(A) \leq 17 < 20$.
Therefore, $\mathcal{CS}(A) \neq \mathbb{R}^{20}$.

This means that there is some vector $\vec{b}$ in $\mathbb{R}^{20}$ that is <u>not</u> in $\mathcal{CS}(A)$.
But, $\vec{b}$ not in $\mathcal{CS}(A)$ means that $A\vec{x} = \vec{b}$ has no solution.

Example

Let A be a 19×56 matrix. Suppose that $A\vec{x} = \vec{b}$ always has a solution. What can we say about the solution spaces to $A\vec{x} = \vec{b}$?

Recall that $\mathcal{NS}(A)$ is a subspace of $\mathbb{R}^{56}$ and $\mathcal{CS}(A)$ is a subspace of $\mathbb{R}^{19}$.

To say that $A\vec{x} = \vec{b}$ always has a solution means that $\mathcal{CS}(A) = \mathbb{R}^{19}$, so $\text{rank}(A) = \dim \mathcal{CS}(A) = 19$.

Also, $\text{rank}(A) + \text{null}(A) = 56$, so $\dim \mathcal{NS}(A) = \text{null}(A) = 56 - 19 = 37$.

Thus $\mathcal{NS}(A)$ is a 37-plane in $\mathbb{R}^{56}$. Remember, the solution spaces to $A\vec{x} = \vec{b}$ are all just translates of $\mathcal{NS}(A)$. Thus every solution space to $A\vec{x} = \vec{b}$ is an *affine* 37-plane in $\mathbb{R}^{56}$.