# The Dimension of a Vector Space

Linear Algebra
MATH 2076

# Bases

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Here $\boldsymbol{t}^i$ denotes the function that satisfies

$$\text{for all numbers } t\,, \quad \boldsymbol{t}^i(t) = t^i.$$

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- $\mathcal{B}$ is a minimal spanning set for $\mathbb{V}$.

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