Dimension, Rank, Nullity

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

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The *dimension* of the trivial vector space $\{\vec{0}\}$ is defined to be 0.

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Examples

Let $\mathbb{U}^{2\times 2}$ and $\mathbb{S}^{2\times 2}$ be the spaces of all upper triangular and all symmetric 2×2 matrices, respectively. Let's find dim $\mathbb{U}^{2\times 2}$ and dim $\mathbb{S}^{2\times 2}$.

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What about upper triangular and symmetric $n \times n$ matrices?

$A = \left[\vec{a}_1 \ \vec{a}_2 \ \ldots \ \vec{a}_n\right]$ an $m \times n$ matrix and $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

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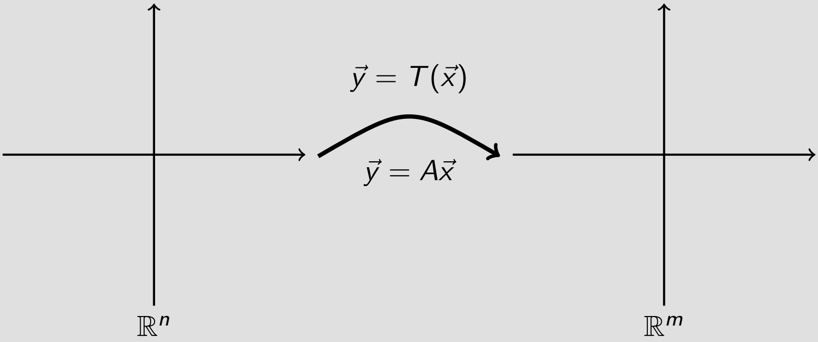

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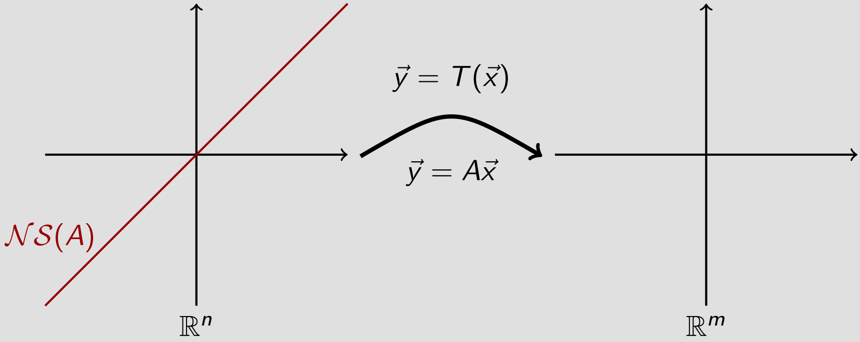

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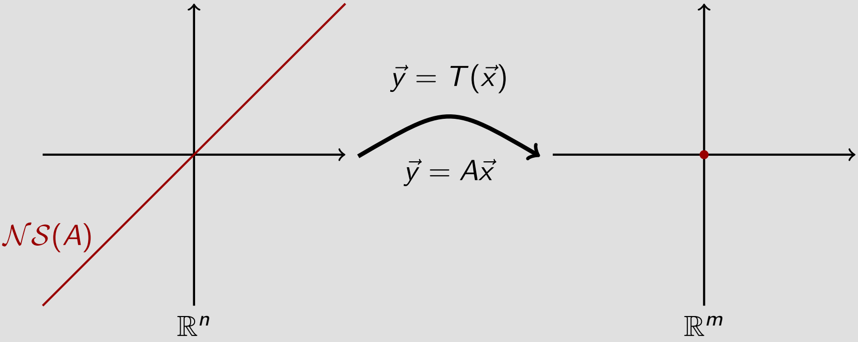

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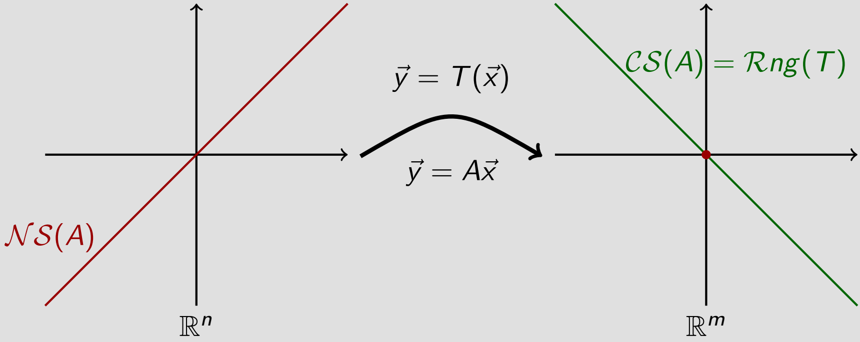

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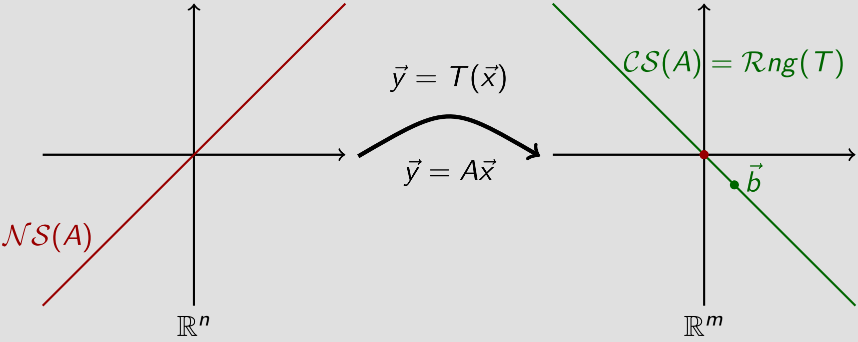

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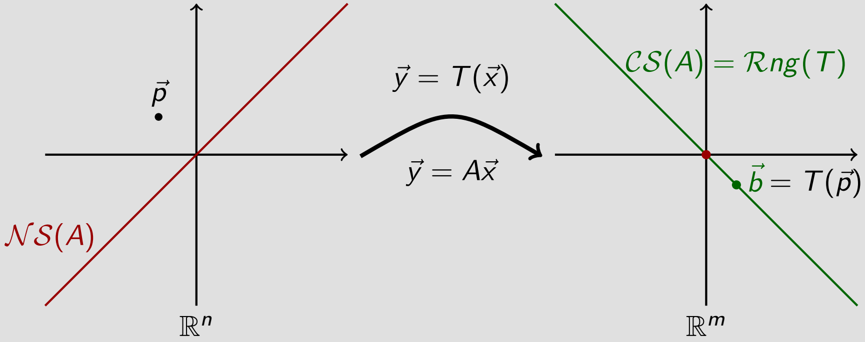

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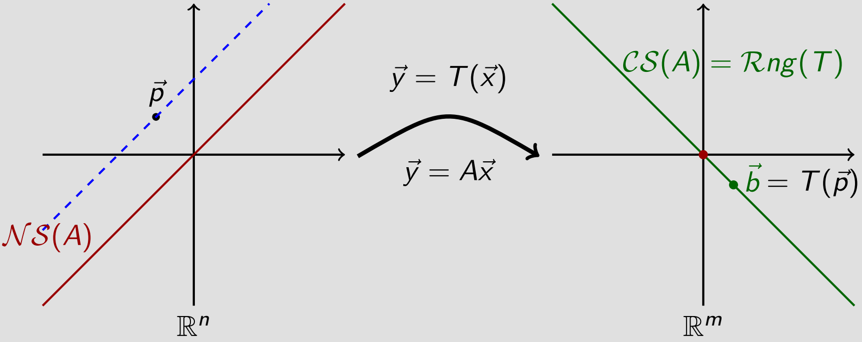

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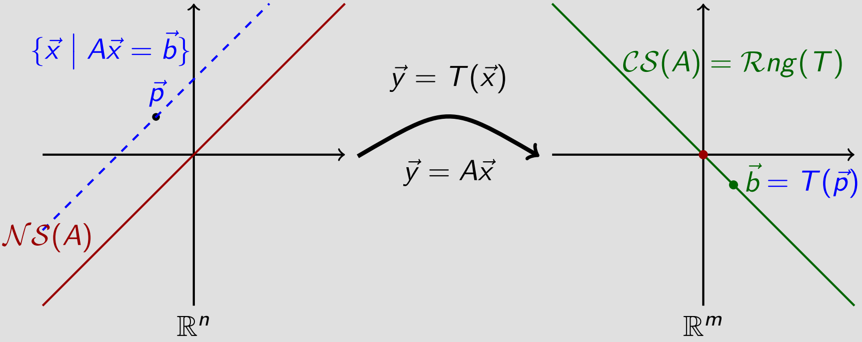

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Find the dimensions of the null space and column space of

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This last fact is called the *Rank-Nullity Theorem*.

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The Rank-Nullity Theorem helps here!

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Thus $\mathcal{NS}(A)$ is a 37-plane in $\mathbb{R}^{56}$. Remember, the solution spaces to $A\vec{x} = \vec{b}$ are all just translates of $\mathcal{NS}(A)$.

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Also, $\text{rank}(A) + \text{null}(A) = 56$, so $\dim \mathcal{NS}(A) = \text{null}(A) = 56 - 19 = 37$.

Thus $\mathcal{NS}(A)$ is a 37-plane in $\mathbb{R}^{56}$. Remember, the solution spaces to $A\vec{x} = \vec{b}$ are all just translates of $\mathcal{NS}(A)$. Thus every solution space to $A\vec{x} = \vec{b}$ is an *affine* 37-plane in $\mathbb{R}^{56}$.