# An Example Using Bases and Coordinates

Linear Algebra
MATH 2076

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Note that $\left[\vec{x}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^p$.

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So, the vectors $\begin{bmatrix} -1 \\ 0 \\ 1 \end{bmatrix}$ and $\begin{bmatrix} 1 \\ 2 \\ 1 \end{bmatrix}$ both lie in $\mathbb{W}$, and these two vectors form a basis for $\mathbb{W}$.

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Recall that $\mathbb{W} = \mathcal{S}pan\{\vec{v}, \vec{w}\}$, so $\mathbb{W}$ is exactly all LCs $s\vec{v} + t\vec{w}$, and then $s, t$ are the $\mathcal{B}$-coordinates of the vector $s\vec{v} + t\vec{w}$.