The Coordinate Mapping Transformation

Linear Algebra
MATH 2076

Coordinates and Coordinate Vectors

Let $\mathcal{B} = \{\vec{a}_1, \ldots, \vec{a}_p\}$ be a basis for a vector space $\mathbb{V}$. Then for each vector $\vec{v}$ in $\mathbb{V}$, there are *unique* scalars $c_1, c_2, \ldots, c_p$ such that

$$\vec{v} = c_1\vec{a}_1 + c_2\vec{a}_2 + \cdots + c_p\vec{a}_p = \sum_{i=1}^{p} c_i\vec{a}_i.$$

We call $c_1, c_2, \ldots, c_p$ the *$\mathcal{B}$-coordinates* of $\vec{v}$ and $\left[\vec{v}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$ is the *$\mathcal{B}$-coordinate vector for $\vec{v}$*.

Note that $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^p$.

Properties of Coordinate Vectors

Again, let $\mathcal{B} = \{\vec{a}_1, \ldots, \vec{a}_p\}$ be a basis for a vector space $\mathbb{V}$. Then:

- for all $\vec{v}, \vec{w}$ in $\mathbb{V}$, $\left[\vec{v} + \vec{w}\right]_{\mathcal{B}} = \left[\vec{v}\right]_{\mathcal{B}} + \left[\vec{w}\right]_{\mathcal{B}}$
- for all scalars s and all $\vec{v}$ in $\mathbb{V}$, $\left[s\vec{v}\right]_{\mathcal{B}} = s\left[\vec{v}\right]_{\mathcal{B}}$

This means that for any vectors $\vec{v}_1, \ldots, \vec{v}_q$ in $\mathbb{V}$, the $\mathcal{B}$-coord vector for any LC of the $\vec{v}_i$'s if the same LC of the $\mathcal{B}$-coord vectors; that is,

$$\left[\sum_{i=1}^{q} s_i \vec{v}_i\right]_{\mathcal{B}} = \sum_{i=1}^{q} s_i \left[\vec{v}_i\right]_{\mathcal{B}}.$$

This is also what tells us that the *$\mathcal{B}$-coord mapping* $\mathbb{V} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^p$ is a linear transformation.

Coordinates for Subspaces of $\mathbb{R}^n$

Suppose $\mathcal{B} = \{\vec{a}_1, \ldots, \vec{a}_p\}$ is a LI set of vectors in $\mathbb{R}^n$. Let $\mathbb{V} = \mathcal{S}pan\,\mathcal{B}$. Then $\mathcal{B}$ is a basis for $\mathbb{V}$.

In this setting, finding coord vectors $\left[\vec{v}\right]_{\mathcal{B}}$ (for $\vec{v}$ in $\mathbb{V}$) is just the problem of solving $A\vec{x} = \vec{v}$ where $A = \left[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_p\right]$.

That is, given a vector $\vec{v}$ in $\mathbb{V}$, $\left[\vec{v}\right]_{\mathcal{B}}$ is just the unique solution to $A\vec{x} = \vec{v}$. This holds because, if $\vec{v} = A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_p\vec{a}_p$, then $x_1, x_2, \ldots, x_p$ are the $\mathcal{B}$-coords of $\vec{v}$.

Again, while $\vec{v}$ is a vector in $\mathbb{R}^n$, $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^p$.

Notice that $\vec{v} = A\left[\vec{v}\right]_{\mathcal{B}}$; that is, multiplication by A changes $\mathcal{B}$-coordinates into standard coordinates. More about this later!

The Coordinate Mapping $\mathbb{V} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^p$

Let $\mathcal{B} = \{\vec{a}_1, \ldots, \vec{a}_p\}$ be a basis for a vector space $\mathbb{V}$. Then each $\vec{v}$ in $\mathbb{V}$ has an associated *$\mathcal{B}$-coordinate vector* $\left[\vec{v}\right]_{\mathcal{B}}$ where $c_1, c_2, \ldots, c_p$ are the *$\mathcal{B}$-coordinates of* $\vec{v}$.
Again, $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^p$.

$$\left[\vec{v}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$$

Now define $\mathbb{V} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^p$ by the formula $\vec{v} \mapsto \left[\vec{v}\right]_{\mathcal{B}}$. This is a LT called the *$\mathcal{B}$-coordinate mapping*.

The inverse of the $\mathcal{B}$-coordinate mapping is easier to understand. This is the linear transformation $\mathbb{R}^p \xrightarrow{T} \mathbb{V}$ given by the formula

$$T(\vec{x}) = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_p\vec{a}_p \quad \text{where} \quad \vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_p \end{bmatrix}.$$

Thus $\vec{x}$ in $\mathbb{R}^p$ is associated to $\vec{v} = T(\vec{x})$ in $\mathbb{V}$ and $\left[\vec{v}\right]_{\mathcal{B}} = \vec{x}$. That is, $\left[T(\vec{x})\right]_{\mathcal{B}} = \left[\vec{v}\right]_{\mathcal{B}} = \vec{x}$.
Can we write T as a matrix transformation?
What if $\mathbb{V}$ is a vector subspace of some $\mathbb{R}^n$?

A 2-plane in $\mathbb{R}^4$

Let $\mathbb{W} = \mathcal{S}pan\{\vec{a}, \vec{b}\}$ where $\vec{a} = \begin{bmatrix} 1 \\ 0 \\ -1 \\ 0 \end{bmatrix}, \vec{b} = \begin{bmatrix} 1 \\ 2 \\ 1 \\ 3 \end{bmatrix}$.

$\mathcal{B} = \{\vec{a}, \vec{b}\}$ is a basis for $\mathbb{W}$

$\mathbb{W}$ is the 2-plane in $\mathbb{R}^4$ consisting of all LCs of $\vec{a}$ and $\vec{b}$; that is, all vectors of the form $s\vec{a} + t\vec{b}$ where s, t are arbitrary scalars.

The *$\mathcal{B}$-coordinate mapping* $\mathbb{W} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^2$ is given by $\vec{w} \mapsto [\vec{w}]_{\mathcal{B}}$. In my opinion, the inverse of the $\mathcal{B}$-coordinate mapping is easier to understand. This is the LT

$$\mathbb{R}^2 \xrightarrow{T} \mathbb{W} \subset \mathbb{R}^4 \quad \text{given by the formula} \quad T\left(\begin{bmatrix} s \\ t \end{bmatrix}\right) = s\vec{a} + t\vec{b}.$$

Evidently, if $\vec{c} = \begin{bmatrix} s \\ t \end{bmatrix}$ and $\vec{w} = T(\vec{c}) = s\vec{a} + t\vec{b}$, then

$$[T(\vec{c})]_{\mathcal{B}} = [\vec{w}]_{\mathcal{B}} = [s\vec{a} + t\vec{b}]_{\mathcal{B}} = \begin{bmatrix} s \\ t \end{bmatrix} = \vec{c}. \text{ Right?}$$

Think of T as transforming the *st*-plane into the 2-plane $\mathbb{W}$ that sits in $\mathbb{R}^4$; T attaches “labels” (that we call $\mathcal{B}$-coords) to each vector in $\mathbb{W}$.

Coordinate Mappings for Subspaces of $\mathbb{R}^n$

Suppose $\mathcal{B} = \{\vec{a}_1, \ldots, \vec{a}_p\}$ is a LI set of vectors in $\mathbb{R}^n$. Let $\mathbb{V} = \mathcal{S}pan\,\mathcal{B}$. Then $\mathcal{B}$ is a basis for $\mathbb{V}$.

In this setting, finding coord vectors $\left[\vec{v}\right]_{\mathcal{B}}$ (for $\vec{v}$ in $\mathbb{V}$) is just the problem of solving $A\vec{x} = \vec{v}$ where $A = \left[\vec{a}_1\ \vec{a}_2 \cdots \vec{a}_p\right]$.

Given $\vec{v}$ in $\mathbb{V}$, $\left[\vec{v}\right]_{\mathcal{B}}$ is the unique solution to $A\vec{x} = \vec{v}$.

That is, if $\vec{c} = \left[\vec{v}\right]_{\mathcal{B}}$, then $A\vec{c} = \vec{v}$, and this is the only such vector with this property. Again, multiplication by A changes $\mathcal{B}$-coordinates into standard coordinates.

Look at the LT $\mathbb{R}^p \xrightarrow{T} \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$. Notice that $\left[T(\vec{x})\right]_{\mathcal{B}} = \vec{x}$. Right?

Write $\vec{c}$ in place of $\vec{x}$ and let $\vec{v} = T(\vec{c}) = A\vec{c}$. Since $A\vec{c} = \vec{v}$, it follows that $\vec{c} = \left[\vec{v}\right]_{\mathcal{B}}$. Right? So, $\left[T(\vec{c})\right]_{\mathcal{B}} = \left[\vec{v}\right]_{\mathcal{B}} = \vec{c}$.

Coordinate Mappings for Subspaces of $\mathbb{R}^n$

We have a LT $\mathbb{R}^p \xrightarrow{T} \mathbb{R}^n$ given by $T(\vec{x}) = A\vec{x}$ where $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 \cdots \vec{a}_p \end{bmatrix}$. Since $\mathbb{V} = \mathcal{S}pan\,\mathcal{B} = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \ldots, \vec{a}_p\} = \mathcal{CS}(A)$, it follows that $\mathcal{R}ng(T) = \mathbb{V}$. Also, $\left[T(\vec{x})\right]_{\mathcal{B}} = \vec{x}$.

Since $\mathcal{B}$ is LI, $A\vec{x} = \vec{0}$ iff $\vec{x} = \vec{0}$, so $\mathcal{K}er(T) = \{\vec{0}\}$. This means T is one-to-one; therefore, $\mathbb{R}^p \xrightarrow{T} \mathcal{R}ng(T) = \mathbb{V}$ has an inverse.

Remember, if $\vec{v} = T(\vec{c}) = A\vec{c}$, then $\vec{c} = \left[\vec{v}\right]_{\mathcal{B}}$. This tells us that the inverse of $\mathbb{R}^p \xrightarrow{T} \mathcal{R}ng(T) = \mathbb{V}$ is the $\mathcal{B}$-coord mapping $\mathbb{V} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^p$. That is, $\mathbb{R}^p \xrightarrow{T} \mathbb{V}$ is the inverse of the $\mathcal{B}$-coord mapping $\mathbb{V} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^p$.

A HyperPlane in $\mathbb{P}_3$ (the space of all cubic polynomials)

Let $\mathbb{W}$ be the space of all polynomials $\boldsymbol{p}$ in $\mathbb{P}_3$ that satisfy $\boldsymbol{p}(2) = 0$.

In the video HypPlaneP3.mp4 we

- Explain why $\mathbb{W}$ is a vector subspace of $\mathbb{P}_3$.
- Find a basis $\mathcal{B}$ for $\mathbb{W}$ and determine the dimension of $\mathbb{W}$.
- Find the $\mathcal{B}$-coordinate vector for $\boldsymbol{p}(t) = (t-1)(t-2)(t-3)$.

We make extensive use of coordinate vectors and related material.

Example—Null Space and Column Space

Find bases for the null space and column space of

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

and determine the corresponding coordinate maps.
Using elem row ops, we find the indicated REF and RREF for A.
Thus columns 1,2,4 are pivot columns for A, so a basis for $\mathcal{CS}(A)$ is given by $\left\{ \begin{bmatrix} 1 \\ 0 \\ 3 \\ 2 \end{bmatrix}, \begin{bmatrix} 2 \\ 1 \\ 6 \\ 4 \end{bmatrix}, \begin{bmatrix} 4 \\ 1 \\ 2 \\ 1 \end{bmatrix} \right\}$ and we see that $\mathcal{CS}(A)$ is a 3-plane in $\mathbb{R}^4$.
Let's focus on $\mathcal{NS}(A)$. So, we need to "solve" $A\vec{x} = \vec{0}$. The free variables are $x_3 = s$, $x_5 = t$; then $x_4 = -2t$, $x_2 = -s + 2t$, $x_1 = -s - t$.

Example—Null Space and Column Space

$$A = \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 3 & 6 & 9 & 2 & -5 \\ 2 & 4 & 6 & 1 & -4 \end{bmatrix} \sim \begin{bmatrix} 1 & 2 & 3 & 4 & 5 \\ 0 & 1 & 1 & 1 & 0 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 1 & 0 & 1 \\ 0 & 1 & 1 & 0 & -2 \\ 0 & 0 & 0 & 1 & 2 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$$

$\mathcal{NS}(A)$ is a vector subspace of $\mathbb{R}^5$. To "find" $\mathcal{NS}(A)$, we solve $A\vec{x} = \vec{0}$.
Free vrbls are $x_3 = s$, $x_5 = t$; then $x_4 = -2t$, $x_2 = -s + 2t$, $x_1 = -s - t$.

Thus $A\vec{x} = \vec{0}$ iff $\vec{x} = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \\ x_4 \\ x_5 \end{bmatrix} = \begin{bmatrix} -s+t \\ -s+2t \\ s \\ -2t \\ t \end{bmatrix} = s \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix} + t \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}$.

So, $\mathcal{NS}(A)$ is a 2-plane in $\mathbb{R}^5$ and the above two vectors form a basis.

Example—Null Space and Column Space

$\mathcal{NS}(A)$ is 2-plane in $\mathbb{R}^5$ with basis $\mathcal{B} = \{\vec{v}_1, \vec{v}_2\}$ where $\vec{v}_1 = \begin{bmatrix} -1 \\ -1 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \vec{v}_2 = \begin{bmatrix} -1 \\ 2 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$

If $\vec{x}$ in $\mathcal{NS}(A)$ is given by $\vec{x} = c_1\vec{v}_1 + c_2\vec{v}_2$,

then $\left[\vec{x}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$ which is a vector in $\mathbb{R}^2$.

The LT $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^5$ given by $T(\vec{c}) = \left[\vec{v}_1 \ \vec{v}_2\right]\vec{c}$ has the property that

$\left[T\left(\begin{bmatrix} c_1 \\ c_2 \end{bmatrix}\right)\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$, and $\mathcal{R}ng(T) = \mathcal{NS}(A) = \mathcal{CS}\Big(\left[\vec{v}_1 \ \vec{v}_2\right]\Big)$.

The inverse of $\mathbb{R}^2 \xrightarrow{T} \mathcal{R}ng(T) = \mathcal{NS}(A) \subset \mathbb{R}^5$ is the $\mathcal{B}$-coordinate mapping $\mathcal{NS}(A) \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^2$ given by $\vec{x} \mapsto \left[\vec{x}\right]_{\mathcal{B}}$.