Bases and Coordinates

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Bases

Let $\mathbb{V}$ be a vector space.

Definition

A set of vectors $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is called a *basis* for $\mathbb{V}$ if and only if

- $\mathcal{B}$ is linearly independent, and
- $\mathcal{B}$ spans $\mathbb{V}$ (i.e., $\mathbb{V} = \mathcal{S}pan(\mathcal{B})$).

So, what are bases useful for? Why do we care about these?

First, since $\mathbb{V} = \mathcal{S}pan(\mathcal{B})$, each vector $\vec{v}$ in $\mathbb{V}$ can be written as a LC of basis vectors. That is, there are scalars scalars $c_1, c_2, \ldots, c_p$ such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \quad \left(\text{more compactly}\,,\ \vec{v} = \sum_{i=1}^{p} c_i\vec{v}_i\right).$$

Next, $\mathcal{B}$ linearly independent says this is the **only** way $\vec{v}$ can be so written.

Bases

Why is $\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p = \sum_{i=1}^{p} c_i\vec{v}_i$ the **only** way that $\vec{v}$ can be written as a LC of vectors in the basis $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$?

To see this, suppose we also have $\vec{v} = \sum_{i=1}^{p} d_i\vec{v}_i$ for some scalars d_i. Then by subtracting $\vec{v} = \sum_{i=1}^{p} d_i\vec{v}_i$ from $\vec{v} = \sum_{i=1}^{p} c_i\vec{v}_i$ we get

$$\sum_{i=1}^{p}(c_i - d_i)\vec{v}_i = \sum_{i=1}^{p} c_i\vec{v}_i - \sum_{i=1}^{p} d_i\vec{v}_i = \vec{v} - \vec{v} = \vec{0}.$$

But since $\mathcal{B}$ is LI, this implies that $c_i = d_i$ for all i. Right?

Using a Basis

Let $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ be a basis for a vector space $\mathbb{V}$. Then for each vector $\vec{v}$ in $\mathbb{V}$, there are *unique* scalars $c_1, c_2, \ldots, c_p$ such that

$$\vec{v} = c_1\vec{v}_1 + c_2\vec{v}_2 + \cdots + c_p\vec{v}_p \quad \left(\text{more compactly}\,,\ \vec{v} = \sum_{i=1}^{p} c_i\vec{v}_i\right).$$

Definition

We call $c_1, c_2, \ldots, c_p$ the *coordinates of $\vec{v}$ relative to $\mathcal{B}$*.

We also call $c_1, c_2, \ldots, c_p$ the *$\mathcal{B}$-coordinates of $\vec{v}$* and $\left[\vec{v}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \\ \vdots \\ c_p \end{bmatrix}$ is the *$\mathcal{B}$-coordinate vector for $\vec{v}$*.

Note that $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^p$.

Example

Let's find a basis for the plane $\mathbb{W}$ in $\mathbb{R}^3$ given by $x + 2y + 3z = 0$. One way to do this is to recognize that $\mathbb{W} = \mathcal{NS}\big([1\ 2\ 3]\big)$ and proceed "as usual". However, it is pretty darn easy to find two LI vectors that span $\mathbb{W}$; these two vectors will form a basis for $\mathbb{W}$.

How can we find *one non-zero* vector in $\mathbb{W}$; i.e., one *non-zero* solution to $x + 2y + 3z = 0$? (We did this sort of thing on the first day of class!) Just set one variable equal to 0, one variable equal to 1, and solve for the third variable; right?

With $z = 0, y = 1$ we get $x = -2$; and with $y = 0, z = 1$ we get $x = -3$. It now follows that a basis for $\mathbb{W}$ is given by

$$\mathcal{B} = \{\vec{w}_1, \vec{w}_2\} \quad \text{where} \quad \vec{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix},\ \vec{w}_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Example

So, a basis for the plane $\mathbb{W}$ (i.e. the soln set to $x + 2y + 3z = 0$) is given by

$$\mathcal{B} = \{\vec{w}_1, \vec{w}_2\} \quad \text{where} \quad \vec{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \ \vec{w}_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Thus every vector $\vec{w}$ in $\mathbb{W}$ can be written in a unique way as $\vec{w} = c_1\vec{w}_1 + c_2\vec{w}_2$ where c_1, c_2 are the $\mathcal{B}$-coordinates of $\vec{w}$, and then

$$\left[\vec{w}\right]_{\mathcal{B}} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad \text{is the } \textit{$\mathcal{B}$-coordinate vector for } \vec{w}.$$

For example,

$$\vec{w} = \begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} = 4\vec{w}_1 - 3\vec{w}_2 \quad \text{is in } \mathbb{W} \text{ and} \quad \left[\vec{w}\right]_{\mathcal{B}} = \begin{bmatrix} 4 \\ -3 \end{bmatrix}.$$

Note that while $\vec{w}$ is in $\mathbb{R}^3$, $\left[\vec{w}\right]_{\mathcal{B}}$ is in $\mathbb{R}^2$.

Using Coordinate Vectors

As above, let $\mathbb{W}$ be the plane in $\mathbb{R}^3$ given by $x + 2y + 3z = 0$, so a basis for $\mathbb{W}$ is given by $\mathcal{B} = \{\vec{w}_1, \vec{w}_2\}$ where

$$\vec{w}_1 = \begin{bmatrix} 2 \\ -1 \\ 0 \end{bmatrix}, \; \vec{w}_2 = \begin{bmatrix} 3 \\ 0 \\ -1 \end{bmatrix}.$$

Now consider the function $\mathbb{W} \xrightarrow{[\cdot]_{\mathcal{B}}} \mathbb{R}^2$ given by the formula $\vec{w} \mapsto \left[\vec{w}\right]_{\mathcal{B}}$. For example, $\begin{bmatrix} -1 \\ -4 \\ 3 \end{bmatrix} \mapsto \begin{bmatrix} 4 \\ -3 \end{bmatrix}$. This LT is called the *$\mathcal{B}$-coordinate mapping*.

The inverse of the $\mathcal{B}$-coordinate mapping is easier to understand. This is the LT $\mathbb{R}^2 \xrightarrow{T} \mathbb{R}^3$ given by the formula

$$T(\vec{c}) = \begin{bmatrix} 2 & 3 \\ -1 & 0 \\ 0 & -1 \end{bmatrix} \vec{c} = c_1 \vec{w}_1 + c_2 \vec{w}_2 \quad \text{where} \quad \vec{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Thus $\vec{c}$ in $\mathbb{R}^2$ is associated to $\vec{w} = T(\vec{c})$ in $\mathbb{W}$ and $\left[\vec{w}\right]_{\mathcal{B}} = \vec{c}$.
Note that $\mathcal{R}ng(T) = \mathbb{W}$.

Example

Let $\vec{a}_1 = \begin{bmatrix} 2 \\ 3 \\ -5 \end{bmatrix}, \vec{a}_2 = \begin{bmatrix} -4 \\ -5 \\ 8 \end{bmatrix}, \vec{u} = \begin{bmatrix} 8 \\ 2 \\ 4 \end{bmatrix}, \vec{v} = \begin{bmatrix} -2 \\ -1 \\ -1 \end{bmatrix}$, $\mathcal{B} = \{\vec{a}_1, \vec{a}_2\}$, and $\mathbb{V} = \mathcal{S}pan\,\mathcal{B}$.

Is $\vec{u}$ or $\vec{v}$ in $\mathbb{V}$, and if so find its $\mathcal{B}$-coordinates.

Can we write $\vec{u}$ (or $\vec{v}$) as a LC of $\vec{a}_1$ and $\vec{a}_2$? Let $A = \left[\vec{a}_1\ \vec{a}_2\right]$.
Can we solve $A\vec{x} = \vec{u}$ (or $A\vec{x} = \vec{v}$)? Look at

$$\left[A \,\middle|\, \vec{u}\ \vec{v}\right] = \left[\begin{array}{rr|rr} 2 & -4 & 8 & -2 \\ 3 & -5 & 2 & -1 \\ -5 & 8 & 4 & -1 \end{array}\right] \sim \left[\begin{array}{rr|rr} 1 & -2 & 4 & -1 \\ 0 & 1 & -10 & 2 \\ 0 & 0 & -9 & 0 \end{array}\right] \sim \left[\begin{array}{rr|rr} 1 & 0 & ? & 3 \\ 0 & 1 & -10 & 2 \\ 0 & 0 & -9 & 0 \end{array}\right]$$

Using elem row ops, we find the indicated REF and RREF for A. See that:
$A\vec{x} = \vec{u}$ has no solutions; $\vec{u}$ not in $\mathbb{V}$, $A\vec{x} = \vec{v}$ has a solution; $\vec{v}$ is in $\mathbb{V}$.

In fact, $A\vec{x} = \vec{v}$ has the solution $\vec{x} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, and therefore $\left[\vec{v}\right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ 2 \end{bmatrix}$, that is, $\vec{v} = 3\vec{a}_1 + 2\vec{a}_2$. Note that $\vec{v}$ is in $\mathbb{R}^3$ whereas $\left[\vec{v}\right]_{\mathcal{B}}$ is in $\mathbb{R}^2$.

Coordinates for Subspaces of $\mathbb{R}^n$

Suppose $\mathcal{B} = \{\vec{a}_1, \ldots, \vec{a}_p\}$ is a LI set of vectors in $\mathbb{R}^n$. Let $\mathbb{V} = \mathcal{S}pan\,\mathcal{B}$. Then $\mathcal{B}$ is a basis for $\mathbb{V}$.

In this setting, finding coord vectors $\left[\vec{v}\right]_{\mathcal{B}}$ (for $\vec{v}$ in $\mathbb{V}$) is just the problem of solving $A\vec{x} = \vec{v}$ where $A = \left[\vec{a}_1 \ \vec{a}_2 \cdots \vec{a}_p\right]$.

That is, given a vector $\vec{v}$ in $\mathbb{V}$, $\left[\vec{v}\right]_{\mathcal{B}}$ is just the unique solution to $A\vec{x} = \vec{v}$. This holds because, if $\vec{v} = A\vec{x} = x_1\vec{a}_1 + x_2\vec{a}_2 + \cdots + x_p\vec{a}_p$, then $x_1, x_2, \ldots, x_p$ are the $\mathcal{B}$-coords of $\vec{v}$.

Again, while $\vec{v}$ is a vector in $\mathbb{R}^n$, $\left[\vec{v}\right]_{\mathcal{B}}$ is a vector in $\mathbb{R}^p$.

Notice that $\vec{v} = A\left[\vec{v}\right]_{\mathcal{B}}$; that is, multiplication by A changes $\mathcal{B}$-coordinates into standard coordinates. More about this later!