Linear Independence and Bases

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Linear Combinations

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The *span* of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$,

$$\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p\}.$$

is the set of *all* LCs of $\vec{v}_1, \vec{v}_2, \ldots \vec{v}_p$.

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When is a single vector $\vec{v}$ LD ? (good quiz question!)

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When is a pair of vectors $\vec{v}_1, \vec{v}_2$ LD ?

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Here we must decide whether or not

$$s_1\vec{v}_1 + \cdots + s_p\vec{v}_p = \vec{0} \implies s_1 = \cdots = s_p = 0.$$

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Therefore, $a\,\mathbf{cos} + b\,\mathbf{sin} = \mathbf{0} \implies a = 0 = b$; so $\mathbf{cos}, \mathbf{sin}$ are LI.

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Now look at **exp** and $\mathbf{exp}^{-1}$; these are the functions given by

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Now easy to see that $a = 0 = b$; so $\mathbf{exp}, \mathbf{exp}^{-1}$ are LI.

Which of these are linearly independent?

$$\cos^2(t), \sin^2(t)$$
$$\cos^2(t), \sin^2(t), \mathbf{1}$$
$$\cos^2(t), \sin^2(t), \cos(2t)$$
$$e^t, t\,e^t$$
$$\sqrt{t}, t\sqrt{t}, t^2\sqrt{t}$$
$$\frac{1}{t-1}, \frac{1}{t+1}$$
$$\frac{1}{t-1}, \frac{1}{t+1}, \frac{1}{t^2-1}$$

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Let $\mathbb{V}$ be a vector space.

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A bunch of vectors $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is called a *basis* for $\mathbb{V}$ if and only if

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Here, as usual,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \ldots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix}.$$

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Recall that $\mathbb{P}_n$ is the vector space of all polynomials of degree n or less.

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