Linear Independence and Bases

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Linear Combinations

Suppose $s_1, s_2, \ldots, s_p$ are scalars and $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are vectors (all in the same vector space $\mathbb{V}$). We call

$$s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p$$

a *linear combination* of the vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$. For example, we always have the *trivial* linear combination

$$0 \cdot \vec{v}_1 + 0 \cdot \vec{v}_2 + \cdots + 0 \cdot \vec{v}_p = \vec{0}.$$

Here we want to know when there is a *non-trivial* LC of $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ that equals $\vec{0}$. This means that $s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p = \vec{0}$, and some scalar $s_j \neq 0$.

Linear Dependence versus Linear Independence

Definition

The vectors $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are *linearly dependent* if there is a *non-trivial* LC of them that equals $\vec{0}$: that is, if there are scalars $s_1, s_2, \ldots, s_p$ so that

$$s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p = \vec{0},$$

and (at least) one of the scalars is *non-zero*.

Linearly dependent vectors carry redundant information.

Definition

Vectors that are *not* LD are said to be *linearly independent*.

Linearly independent vectors carry NO redundant information.

Linear Independence

Definition

Vectors that are *not* LD are said to be *linearly independent*.

How do we show linear independence?

For vectors in a Euclidean space, we can form a matrix A and determine whether or not there are non-trivial solutions to $A\vec{x} = \vec{0}$. Review the Invertible Matrix Theorem!

But what about more general vectors $\vec{v}_1, \ldots, \vec{v}_p$?
(For example, if the $\vec{v}_i$ are all functions?)

Here we must decide whether or not

$$s_1\vec{v}_1 + \cdots + s_p\vec{v}_p = \vec{0} \implies s_1 = \cdots = s_p = 0.$$

Example—vectors in $\mathbb{F}$

Recall that $\mathbb{F} = \mathbb{F}(\mathbb{R} \to \mathbb{R}) = \{\text{all } \boldsymbol{f} \text{ with } \mathbb{R} \xrightarrow{\boldsymbol{f}} \mathbb{R}\}$ is a vector space with the usual ways of adding and multiplying by scalars.
Here we examine various pairs (and triples) of vectors in $\mathbb{F}$.

Look at **cos** and **sin**. Are these LD or LI? Is there a *non-trivial* LC $a\,\mathbf{cos} + b\,\mathbf{sin}$ that equals zero (the zero function), or does $a\,\mathbf{cos} + b\,\mathbf{sin} = \mathbf{0} \implies a = 0 = b$?

What does $a\,\mathbf{cos} + b\,\mathbf{sin} = \mathbf{0}$ even mean? Here, of course, a and b are scalars. What is $\mathbf{0}$? It is the *zero function*, right?

So, $a\,\mathbf{cos} + b\,\mathbf{sin} = \mathbf{0}$ means that for all t,

$$a\,\cos(t) + b\,\sin(t) = 0.$$

Let's plug in some values for t. Try $t = 0$ and then $t = \pi/2$. We get

$$a \cdot 1 + b \cdot 0 = 0, \text{ so } a = 0 \quad \text{and then} \quad a \cdot 0 + b \cdot 1 = 0, \text{ so } b = 0.$$

Therefore, $a\,\mathbf{cos} + b\,\mathbf{sin} = \mathbf{0} \implies a = 0 = b$; so $\mathbf{cos}, \mathbf{sin}$ are LI.

Example—vectors in $\mathbb{F}$

Now look at $\mathbf{exp}$ and $\mathbf{exp}^{-1}$; these are the functions given by

$$\mathbf{exp}(t) = e^t \quad \text{and} \quad \mathbf{exp}^{-1}(t) = e^{-t}.$$

Are these LD or LI?
Is there a *non-trivial* LC $a\,\mathbf{exp} + b\,\mathbf{exp}^{-1}$ that equals zero, or does $a\,\mathbf{exp} + b\,\mathbf{exp}^{-1} = \mathbf{0} \implies a = 0 = b$?
Again, $a\,\mathbf{exp} + b\,\mathbf{exp}^{-1} = \mathbf{0}$ means that for all t,

$$a\,e^t + b\,e^{-t} = 0.$$

Can plug in values for t. Try $t = 0$ and then $t = \ln 2$. We get

$$a + b = 0 \quad \text{and then} \quad 2a + \frac{1}{2}\,b = 0.$$

Can show this SLE has unique soln $a = 0 = b$; so $\mathbf{exp}, \mathbf{exp}^{-1}$ are LI.

Here's alternative method. Start with

$$a\,e^t + b\,e^{-t} = 0 \quad \text{and then differentiate to get} \quad a\,e^t - b\,e^{-t} = 0.$$

Now easy to see that $a = 0 = b$; so $\mathbf{exp}, \mathbf{exp}^{-1}$ are LI.

Which of these are linearly independent?

$$\cos^2(t), \sin^2(t)$$

$$\cos^2(t), \sin^2(t), \mathbf{1}$$

$$\cos^2(t), \sin^2(t), \cos(2t)$$

$$e^t, t\, e^t$$

$$\sqrt{t}, t\sqrt{t}, t^2\sqrt{t}$$

$$\frac{1}{t-1}, \frac{1}{t+1}$$

$$\frac{1}{t-1}, \frac{1}{t+1}, \frac{1}{t^2-1}$$

Bases for Vector Spaces

Let $\mathbb{V}$ be a vector space.

Definition

A set of vectors $\mathcal{B} = \{\vec{v}_1, \ldots, \vec{v}_p\}$ is called a *basis* for $\mathbb{V}$ if and only if

- $\mathcal{B}$ is linearly independent, and
- $\mathcal{B}$ spans $\mathbb{V}$ (i.e., $\mathbb{V} = \mathcal{S}pan(\mathcal{B})$).

Example (Standard Basis for $\mathbb{R}^n$)

The set $\mathcal{S} = \{\vec{e}_1, \ldots, \vec{e}_n\}$ is the *standard basis* for $\mathbb{R}^n$.

Here, as usual,

$$\vec{e}_1 = \begin{bmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \quad \vec{e}_2 = \begin{bmatrix} 0 \\ 1 \\ \vdots \\ 0 \end{bmatrix}, \quad \ldots, \quad \vec{e}_n = \begin{bmatrix} 0 \\ 0 \\ \vdots \\ 1 \end{bmatrix} \quad \text{and} \quad \vec{x} = \sum_{i=1}^{n} x_i \vec{e}_i \,.$$

Basis for $\mathbb{P}_n$

Recall that $\mathbb{P}_n$ is the vector space of all polynomials of degree n or less.

Example (Standard Basis for $\mathbb{P}_n$)

The set $\mathcal{P} = \{\mathbf{1}, \boldsymbol{t}, \boldsymbol{t^2}, \dots, \boldsymbol{t^n}\}$ is the *standard basis* for $\mathbb{P}_n$.

Here $\boldsymbol{t^i}$ denotes the function that satisfies

$$\text{for all real numbers } t\,, \quad \boldsymbol{t^i}(t) = t^i.$$

We know that each poly $\boldsymbol{p}$ in $\mathbb{P}_n$ is a function of the form

$$\boldsymbol{p}(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n\,,$$

which means that

$$\boldsymbol{p} = c_0\mathbf{1} + c_1\boldsymbol{t} + c_2\boldsymbol{t^2} + \cdots + c_n\boldsymbol{t^n}$$

so $\mathcal{P}$ does indeed span $\mathbb{P}_n$, right? Why is $\mathcal{P}$ LI?

Equivalent Views for a Basis

Let $\mathbb{V}$ be a vector space. The following are equivalent:

- $\mathcal{B}$ is a basis for $\mathbb{V}$.
- $\mathcal{B}$ is a minimal spanning set for $\mathbb{V}$.
- $\mathcal{B}$ is a maximal linearly independent set in $\mathbb{V}$.