Linear Transformations Between Vector Spaces

Linear Algebra
MATH 2076

University of Cincinnati

Transformations $\mathbb{V} \xrightarrow{T} \mathbb{W}$

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces, and let $\mathbb{V} \xrightarrow{T} \mathbb{W}$ be a function (aka, a *transformation*).

Here $\mathbb{V}$ is the *domain* of T (where the input variables $\vec{x}$ live) and $\mathbb{W}$ is the *codomain* of T (where the resulting output $\vec{y} = T(\vec{x})$ lives).

For each $\vec{x}$ in $\mathbb{V}$, $\vec{y} = T(\vec{x})$ is called the *image* of $\vec{x}$. If $\mathbb{S}$ is a bunch of vectors in $\mathbb{V}$ (i.e., $\mathbb{S}$ is a *subset* of $\mathbb{V}$), then

$$T(\mathbb{S}) = \left\{ \text{all images } T(\vec{x}) \text{ where } \vec{x} \text{ is in } \mathbb{S} \right\}$$

is called the *T-image of $\mathbb{S}$*; this is a subset of $\mathbb{W}$.

The *range* of T is the image of all of $\mathbb{V}$, i.e., $\boxed{\mathcal{R}ng(T) = T(\mathbb{V})}$; this is the set of <u>all</u> images $T(\vec{x})$, and $\mathcal{R}ng(T)$ is a subset of the codomain of T.

An important question is to know which vectors $\vec{b}$ are in the range of T. That is: Given $\vec{b}$ in $\mathbb{W}$, when can we find an $\vec{x}$ in $\mathbb{V}$ with $T(\vec{x}) = \vec{b}$?

Linear Transformations $\mathbb{V} \xrightarrow{T} \mathbb{W}$

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces, and let $\mathbb{V} \xrightarrow{T} \mathbb{W}$ be a function (aka, a *transformation*).

We call T a *linear transformation* provided

$$T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})$$

and

$$T(s\vec{v}) = sT(\vec{v})$$

for all $\vec{u}, \vec{v}$ in $\mathbb{R}^n$ and all scalars s.

Note that for **any** LT T we *always have* $T(\vec{0}) = \vec{0}$. Right? Why?

Recall that *every* LT between two Euclidean spaces is a matrix transformation. In general?

Properties of Linear Transformations

Let $\mathbb{V} \xrightarrow{T} \mathbb{W}$ be a linear transformation. Then $\boxed{T(\vec{0}) = \vec{0}}$.

More importantly, T *preserves* all linear combinations; i.e., the T-image of a LC of vectors $\vec{v}_j$ is a LC of $T(\vec{v}_j)$ using the same scalars. That is,

$$T(s_1\vec{v}_1 + s_2\vec{v}_2 + \cdots + s_p\vec{v}_p) = s_1 T(\vec{v}_1) + s_2 T(\vec{v}_2) + \cdots + s_p T(\vec{v}_p)$$

or more simply—using "summation" notation—

$$T\left(\sum_{j=1}^{p} s_j \vec{v}_j\right) = \sum_{j=1}^{p} s_j T(\vec{v}_j).$$

This is called the *linearity principle*.

Range and Kernel of a Linear Transformation

Let $\mathbb{V} \xrightarrow{T} \mathbb{W}$ be a linear transformation. The *range* of T is the image of all of $\mathbb{V}$, i.e.,

$$\mathcal{R}ng(T) = T(\mathbb{V}) = \{\, T(\vec{v}) \mid \vec{v} \text{ in } \mathbb{V} \,\};$$

this is a vector subspace of $\mathbb{W}$.

The *kernel* of T is the pre-image of $\vec{0}$, i.e.,

$$\mathcal{K}er(T) = \{\, \vec{v} \text{ in } \mathbb{V} \mid T(\vec{v}) = \vec{0} \,\};$$

this is a vector subspace of $\mathbb{V}$.

When $\mathbb{V}$ and $\mathbb{W}$ are Euclidean spaces, and T is mult by a matrix A

$$\mathcal{K}er(T) = \mathcal{NS}(A) \quad \text{and} \quad \mathcal{R}ng(T) = \mathcal{CS}(A).$$

Examples of Linear Transformations

Let A be a fixed $m \times n$ matrix. Define $\mathbb{R}^{n\times p} \xrightarrow{T} \mathbb{R}^{m\times p}$ by $T(X) = AX$; T is a linear transformation. Without knowing more about A, we cannot say much about $\mathcal{K}er(T)$ or $\mathcal{R}ng(T)$.

We can also define $\mathbb{R}^{p\times m} \xrightarrow{S} \mathbb{R}^{p\times n}$ by $T(X) = XA$; S is an LT, but without knowing A, we can't "find" $\mathcal{K}er(S)$ or $\mathcal{R}ng(S)$.

Now define $\mathbb{R}^{m\times n} \xrightarrow{T} \mathbb{R}^{n\times m}$ by $T(X) = X^T$; recall that X^T is the *transpose* of the matrix X given by $\left[X^T\right]_{ij} = \left[X\right]_{ji}$. Then T is an LT. What are $\mathcal{K}er(T)$ and $\mathcal{R}ng(T)$? This is easy, right?

Next, define $\mathbb{R}^{n\times n} \xrightarrow{S} \mathbb{R}^{n\times n}$ by $T(X) = X - X^T$; you should check that S is an LT. What are $\mathcal{K}er(S)$ and $\mathcal{R}ng(S)$?

If $S(X) = 0$, then $X = X^T$; so $\mathcal{K}er(S)$ is the subspace of all *symmetric* $n \times n$ matrices.
HW: Show $\mathcal{R}ng(S)$ is the subspace of all *skew-symmetric* $n \times n$ matrices.

More Examples of Linear Transformations

Recall that $\mathbb{P}$ is the vector space of all polynomials. Thus $\boldsymbol{p}$ is in $\mathbb{P}$ if and only if $\boldsymbol{p}$ is a function of the form

$$\boldsymbol{p}(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n.$$

Here $c_0, c_1, \ldots, c_n$ are constants, called the *coefficients* of the polynomial $\boldsymbol{p}$, and when $c_n \neq 0$ we say that $\boldsymbol{p}$ has *degree n*. (By definition, the zero polynomial has degree zero.)

Define $\mathbb{P} \xrightarrow{T} \mathbb{R}^3$ by $T(\boldsymbol{p}) = \begin{bmatrix} \boldsymbol{p}(1) \\ \boldsymbol{p}(2) \\ \boldsymbol{p}(3) \end{bmatrix}$.

Check that T is an LT.
What are $\mathcal{K}er(T)$ & $\mathcal{R}ng(T)$?

This is not too hard; please try to find these yourself before we do it.

Clearly, $\mathcal{R}ng(T) = \mathbb{R}^3$. Right? Why?? Also, a poly $\boldsymbol{p}$ is in $\mathcal{K}er(T)$ iff $\boldsymbol{p}$ has zeroes at each of $t = 1, 2, 3$; so $\boldsymbol{p}$ is in $\mathcal{K}er(T)$ iff $\deg(\boldsymbol{p}) \geq 3$ and $\boldsymbol{p}$ can be factored as

$$\boldsymbol{p}(t) = (t-1)(t-2)(t-3)\boldsymbol{q}(t).$$

More Examples of Linear Transformations

Recall that $\mathbb{P}$ is the vector space of all polynomials. Define $\mathbb{P} \xrightarrow{D} \mathbb{P}$ by differentiation: $D(\boldsymbol{p}) = \boldsymbol{p}'$. By Calculus, D is an LT.

What are $\mathcal{K}er(D)$ and $\mathcal{R}ng(D)$?

What is $\mathcal{K}er(D)$? This is just the "trivial" subspace of all *constant* polynomials, right? So, $\mathcal{K}er(D) = \mathcal{S}pan\{\mathbf{1}\}$.

What is $\mathcal{R}ng(D)$? Which polys are the derivative of some other poly? Let $\boldsymbol{q}$ be a poly, say

$$\boldsymbol{q}(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n.$$

Can we find some other poly $\boldsymbol{p}$ so that $\boldsymbol{p}' = D(\boldsymbol{p}) = \boldsymbol{q}$? Sure! We just anti-differentiate, right? The poly

$$\boldsymbol{p}(t) = c_0 t + \frac{c_1}{2} t^2 + \frac{c_2}{3} t^3 + \cdots + \frac{c_n}{n+1} t^{n+1}$$

has the property that $\boldsymbol{p}' = D(\boldsymbol{p}) = \boldsymbol{q}$. This means that $\mathcal{R}ng(D) = \mathbb{P}$.

And More Examples of Linear Transformations

Recall that $\mathbb{P}$ is the vector space of all polynomials. Define $\mathbb{P}_n \xrightarrow{D} \mathbb{P}_n$ by differentiation: $D(\boldsymbol{p}) = \boldsymbol{p}'$. By Calculus, D is an LT.

This is clearly similar to the above example, but notice that here we are taking the domain and codomain of D to be $\mathbb{P}_n$ instead of $\mathbb{P}$.

What are $\mathcal{K}er(D)$ and $\mathcal{R}ng(D)$? This is not too hard; please try to find these yourself before we give the answers.

Again, $\mathcal{K}er(D)$ is the "trivial" subspace of all *constant* polynomials, right?

What about $\mathcal{R}ng(D)$?

HW: Show that $\mathcal{R}ng(D) = \mathbb{P}_{n-1}$.

One More Example of a Linear Transformation

Recall that $\mathbb{P}$ is the vector space of all polynomials. Define $\mathbb{P} \xrightarrow{S} \mathbb{P}$ by "multiplication": i.e., $\boldsymbol{q} = S(\boldsymbol{p})$ is the polynomial given by $\boldsymbol{q}(t) = t\,\boldsymbol{p}(t)$. You should check that S is an LT.

What are $\mathcal{K}er(S)$ and $\mathcal{R}ng(S)$?

Remember, $\mathcal{K}er(S)$ is all polys $\boldsymbol{p}$ with $S(\boldsymbol{p}) = \mathbf{0}$. Notice that the poly $\mathbf{1}$ has $S(\mathbf{1}) = \boldsymbol{t}$, right? So, every constant non-zero poly has a non-zero S image. We now see that $\mathcal{K}er(S) = \{\mathbf{0}\}$.

What about $\mathcal{R}ng(D)$? Notice that every S image $\boldsymbol{q} = S(\boldsymbol{p})$ has $\boldsymbol{q}(0) = 0$. Using this we see that $\mathcal{R}ng(S)$ consists of all polys $\boldsymbol{q}$ with "no constant term"; i.e.,

$$\mathcal{R}ng(S) = \Big\{\boldsymbol{q} \;\Big|\; \boldsymbol{q}(t) = b_1 t + b_2 t^2 + \cdots + b_n t^n\Big\}.$$

HW: Show that $DS - SD = I$ where D is "differentiation" (as described 2 slides above) and I is the identity transformation $I(\boldsymbol{p}) = \boldsymbol{p}$.