Transformations $\mathbb{V} \xrightarrow{T} \mathbb{W}$

Let $\mathbb{V}$ and $\mathbb{W}$ be vector spaces, and let $\mathbb{V} \xrightarrow{T} \mathbb{W}$ be a function (aka, a transformation).

Here $\mathbb{V}$ is the domain of $T$ (where the input variables $\vec{x}$ live) and $\mathbb{W}$ is the codomain of $T$ (where the resulting output $\vec{y} = T(\vec{x})$ lives).

For each $\vec{x}$ in $\mathbb{V}$, $\vec{y} = T(\vec{x})$ is called the image of $\vec{x}$. If $S$ is a bunch of vectors in $\mathbb{V}$ (i.e., $S$ is a subset of $\mathbb{V}$), then

$$T(S) = \{ \text{all images } T(\vec{x}) \text{ where } \vec{x} \text{ is in } S \}$$

is called the $T$-image of $S$; this is a subset of $\mathbb{W}$.

The range of $T$ is the image of all of $\mathbb{V}$, i.e., $\text{Rng}(T) = T(\mathbb{V})$; this is the set of all images $T(\vec{x})$, and $\text{Rng}(T)$ is a subset of the codomain of $T$.

An important question is to know which vectors $\vec{b}$ are in the range of $T$. That is: Given $\vec{b}$ in $\mathbb{W}$, when can we find an $\vec{x}$ in $\mathbb{V}$ with $T(\vec{x}) = \vec{b}$?
Linear Transformations \[ \mathbb{V} \overset{T}{\rightarrow} \mathbb{W} \]

Let \( \mathbb{V} \) and \( \mathbb{W} \) be vector spaces, and let \( \mathbb{V} \overset{T}{\rightarrow} \mathbb{W} \) be a function (aka, a transformation).

We call \( T \) a **linear transformation** provided

\[
T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v})
\]

and

\[
T(s\vec{v}) = sT(\vec{v})
\]

for all \( \vec{u}, \vec{v} \) in \( \mathbb{R}^n \) and all scalars \( s \).

Note that for any LT \( T \) we always have \( T(\vec{0}) = \vec{0} \). Right? Why?

Recall that every LT between two Euclidean spaces is a matrix transformation. In general?
Properties of Linear Transformations

Let $\mathbb{V} \xrightarrow{T} \mathbb{W}$ be a linear transformation. Then $T(\vec{0}) = \vec{0}$.

More importantly, $T$ preserves all linear combinations; i.e., the $T$-image of a LC of vectors $\vec{v}_j$ is a LC of $T(\vec{v}_j)$ using the same scalars. That is,

$$T(s_1 \vec{v}_1 + s_2 \vec{v}_2 + \cdots + s_p \vec{v}_p) = s_1 T(\vec{v}_1) + s_2 T(\vec{v}_2) + \cdots + s_p T(\vec{v}_p)$$

or more simply—using “summation” notation—

$$T \left( \sum_{j=1}^{p} s_j \vec{v}_j \right) = \sum_{j=1}^{p} s_j T(\vec{v}_j).$$

This is called the **linearity principle**.
Let $\mathbb{V} \xrightarrow{T} \mathbb{W}$ be a linear transformation. The **range** of $T$ is the image of all of $\mathbb{V}$, i.e.,

$$\text{Rng}(T) = T(\mathbb{V}) = \{ T(\vec{v}) \mid \vec{v} \text{ in } \mathbb{V} \};$$

this is a vector subspace of $\mathbb{W}$.

The **kernel** of $T$ is the pre-image of $\vec{0}$, i.e.,

$$\text{Ker}(T) = \{ \vec{v} \text{ in } \mathbb{V} \mid T(\vec{v}) = \vec{0} \};$$

this is a vector subspace of $\mathbb{V}$.

When $\mathbb{V}$ and $\mathbb{W}$ are Euclidean spaces, and $T$ is mult by a matrix $A$

$$\text{Ker}(T) = \text{NS}(A) \quad \text{and} \quad \text{Rng}(T) = \text{CS}(A).$$
Examples of Linear Transformations

Let $A$ be a fixed $m \times n$ matrix. Define $\mathbb{R}^{n \times p} \xrightarrow{T} \mathbb{R}^{m \times p}$ by $T(X) = AX$; $T$ is a linear transformation. Without knowing more about $A$, we cannot say much about $\ker(T)$ or $\text{Rng}(T)$.

We can also define $\mathbb{R}^{p \times m} \xrightarrow{S} \mathbb{R}^{p \times n}$ by $T(X) = XA$; $S$ is an LT, but without knowing $A$, we can’t “find” $\ker(S)$ or $\text{Rng}(S)$.

Now define $\mathbb{R}^{m \times n} \xrightarrow{T} \mathbb{R}^{n \times m}$ by $T(X) = X^T$; recall that $X^T$ is the transpose of the matrix $X$ given by $[X^T]_{ij} = [X]_{ji}$. Then $T$ is an LT. What are $\ker(T)$ and $\text{Rng}(T)$? This is easy, right?

Next, define $\mathbb{R}^{n \times n} \xrightarrow{S} \mathbb{R}^{n \times n}$ by $T(X) = X - X^T$; you should check that $S$ is an LT. What are $\ker(S)$ and $\text{Rng}(S)$?

If $S(X) = 0$, then $X = X^T$; so $\ker(S)$ is the subspace of all symmetric $n \times n$ matrices.

HW: Show $\text{Rng}(S)$ is the subspace of all skew-symmetric $n \times n$ matrices.
More Examples of Linear Transformations

Recall that \( \mathbb{P} \) is the vector space of all polynomials. Thus \( p \) is in \( \mathbb{P} \) if and only if \( p \) is a function of the form

\[
p(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n.
\]

Here \( c_0, c_1, \ldots, c_n \) are constants, called the coefficients of the polynomial \( p \), and when \( c_n \neq 0 \) we say that \( p \) has degree \( n \). (By definition, the zero polynomial has degree zero.)

Define \( \mathbb{P} \xrightarrow{T} \mathbb{R}^3 \) by \( T(p) = \begin{bmatrix} p(1) \\ p(2) \\ p(3) \end{bmatrix} \). Check that \( T \) is an LT. What are \( \text{Ker}(T) \) & \( \text{Rng}(T) \)?

This is not too hard; please try to find these yourself before we do it.

Clearly, \( \text{Rng}(T) = \mathbb{R}^3 \). Right? Why?? Also, a poly \( p \) is in \( \text{Ker}(T) \) iff \( p \) has zeroes at each of \( t = 1, 2, 3 \); so \( p \) is in \( \text{Ker}(T) \) iff \( \deg(p) \geq 3 \) and \( p \) can be factored as

\[
p(t) = (t - 1)(t - 2)(t - 3)q(t).
\]
More Examples of Linear Transformations

Recall that \( \mathbb{P} \) is the vector space of all polynomials. Define \( \mathbb{P} \xrightarrow{D} \mathbb{P} \) by differentiation: \( D(p) = p' \). By Calculus, \( D \) is an LT.

What are \( \text{Ker}(D) \) and \( \text{Rng}(D) \)?

What is \( \text{Ker}(D) \)? This is just the “trivial” subspace of all constant polynomials, right? So, \( \text{Ker}(D) = \text{Span}\{1\} \).

What is \( \text{Rng}(D) \)? Which polys are the derivative of some other poly? Let \( q \) be a poly, say

\[
q(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n.
\]

Can we find some other poly \( p \) so that \( p' = D(p) = q \)? Sure! We just anti-differentiate, right? The poly

\[
p(t) = c_0 t + \frac{c_1}{2} t^2 + \frac{c_2}{3} t^3 + \cdots + \frac{c_n}{n + 1} t^{n+1}
\]

has the property that \( p' = D(p) = q \). This means that \( \text{Rng}(D) = \mathbb{P} \).
Recall that $\mathbb{P}$ is the vector space of all polynomials. Define $\mathbb{P}_n \xrightarrow{D} \mathbb{P}_n$ by differentiation: $D(p) = p'$. By Calculus, $D$ is an LT.

This is clearly similar to the above example, but notice that here we are taking the domain and codomain of $D$ to be $\mathbb{P}_n$ instead of $\mathbb{P}$.

What are $\ker(D)$ and $\text{Rng}(D)$? This is not too hard; please try to find these yourself before we give the answers.

Again, $\ker(D)$ is the “trivial” subspace of all constant polynomials, right?

What about $\text{Rng}(D)$?

HW: Show that $\text{Rng}(D) = \mathbb{P}_{n-1}$. 
One More Example of a Linear Transformation

Recall that \( \mathbb{P} \) is the vector space of all polynomials. Define \( \mathbb{P} \xrightarrow{S} \mathbb{P} \) by “multiplication”: i.e., \( q = S(p) \) is the polynomial given by \( q(t) = t \cdot p(t) \). You should check that \( S \) is an LT.

What are \( \text{Ker}(S) \) and \( \text{Rng}(S) \)?

Remember, \( \text{Ker}(S) \) is all polynomials \( p \) with \( S(p) = 0 \). Notice that the poly \( 1 \) has \( S(1) = t \), right? So, every constant non-zero poly has a non-zero \( S \) image. We now see that \( \text{Ker}(S) = \{0\} \).

What about \( \text{Rng}(D) \)? Notice that every \( S \) image \( q = S(p) \) has \( q(0) = 0 \). Using this we see that \( \text{Rng}(S) \) consists of all polynomials \( q \) with “no constant term”; i.e.,

\[
\text{Rng}(S) = \left\{ q \mid q(t) = b_1 t + b_2 t^2 + \cdots + b_n t^n \right\}.
\]

HW: Show that \( DS - SD = I \) where \( D \) is “differentiation” (as described 2 slides above) and \( I \) is the identity transformation \( I(p) = p \).