Subspaces of Euclidean Space $\mathbb{R}^n$

Linear Algebra
MATH 2076

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Example (Basic Vector SubSpace)

For any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in a vector space $\mathbb{V}$, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is the vector subspace of $\mathbb{V}$ *spanned* by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

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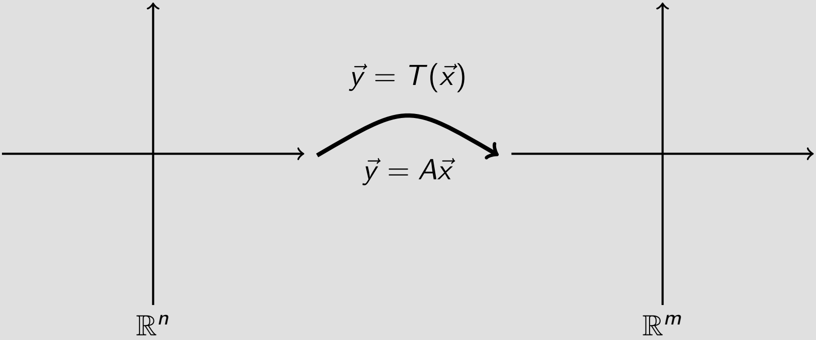

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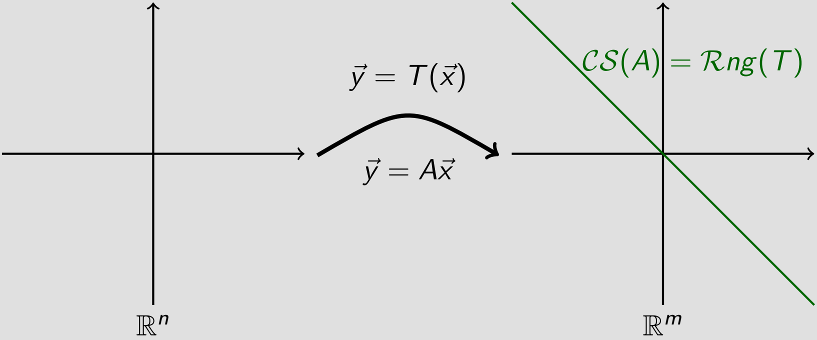

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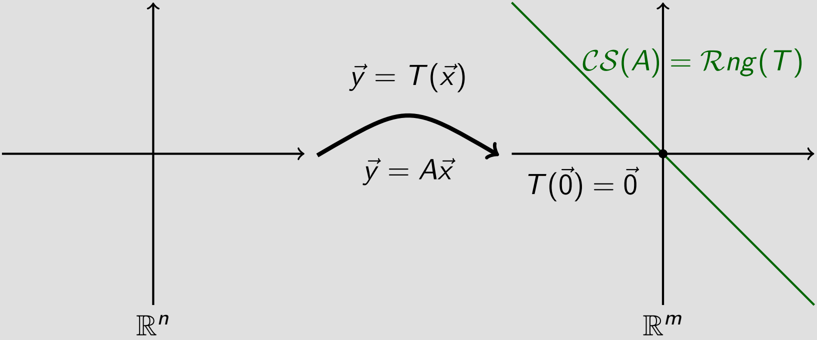

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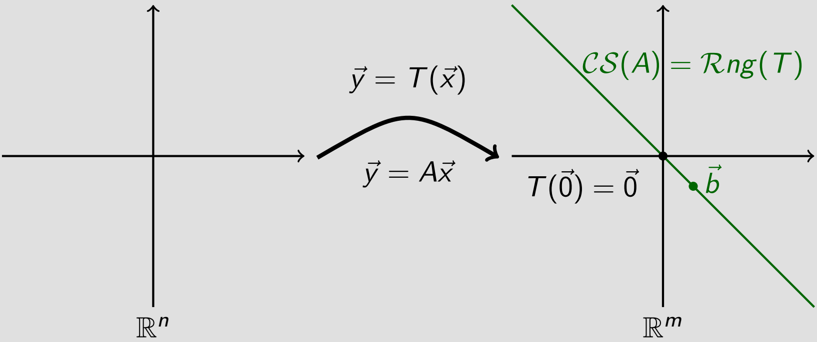

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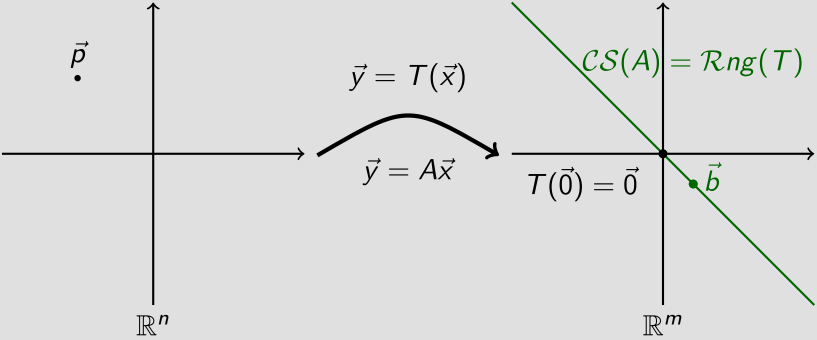

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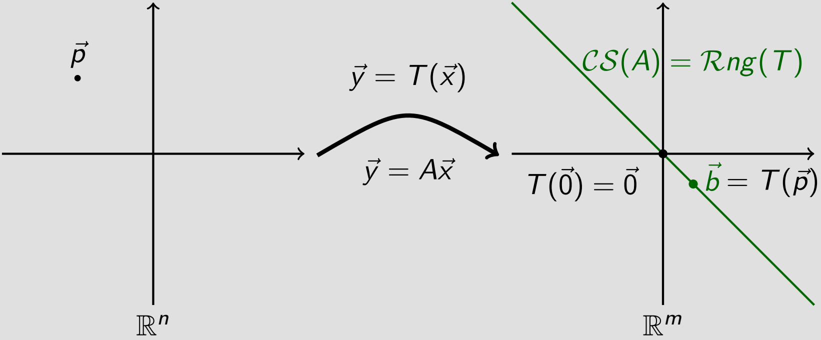

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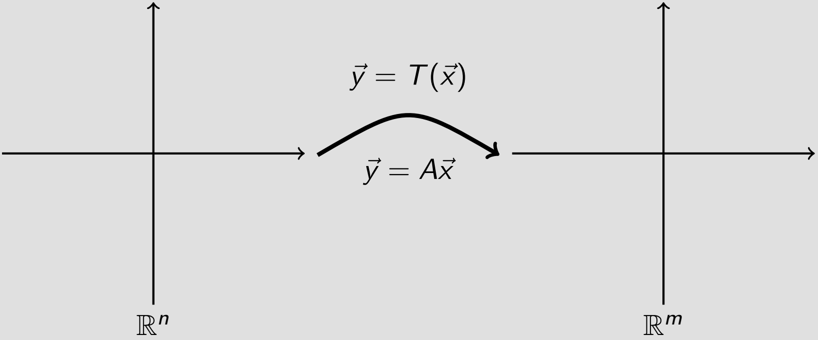

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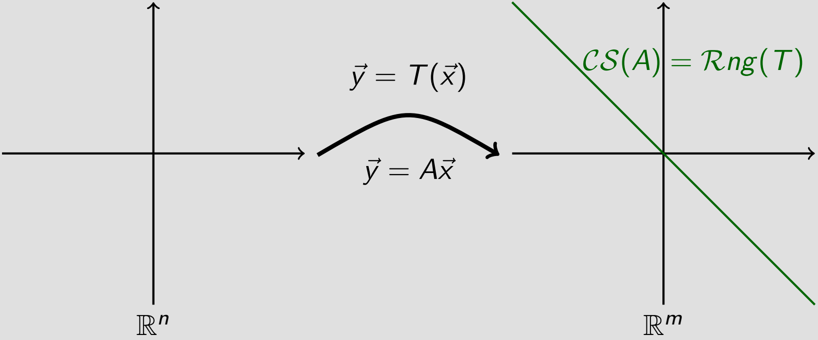

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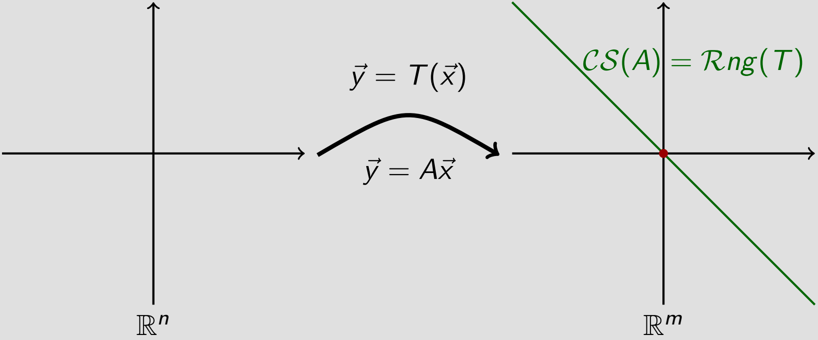

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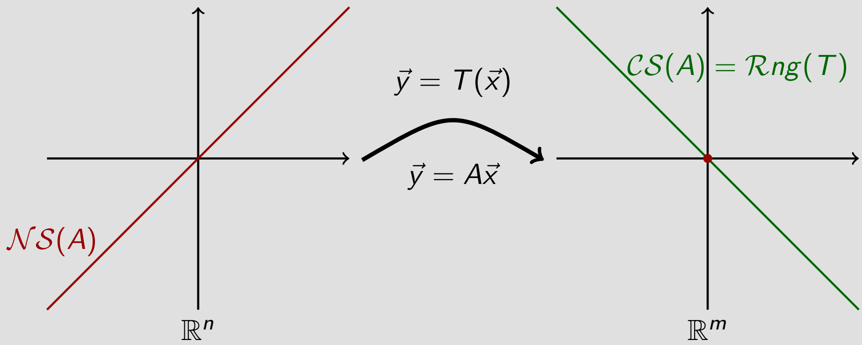

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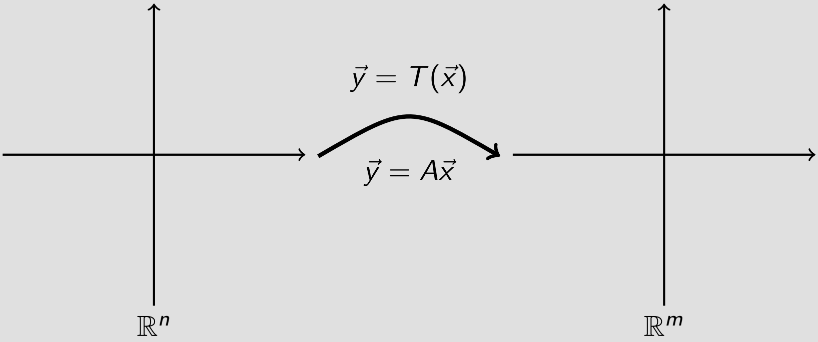

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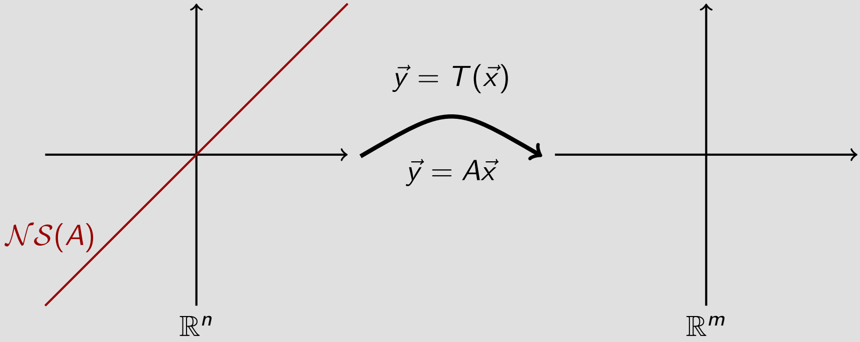

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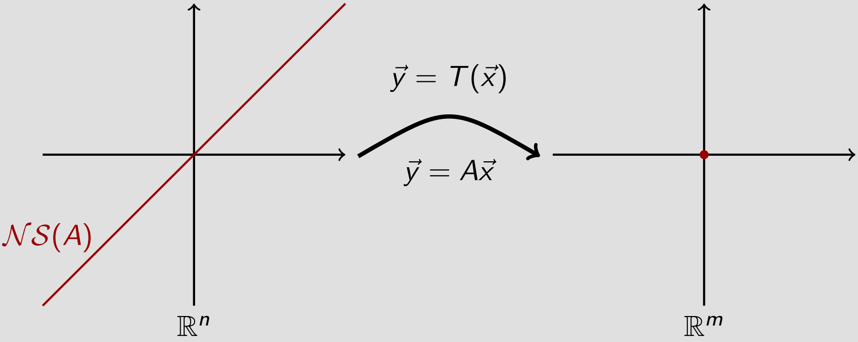

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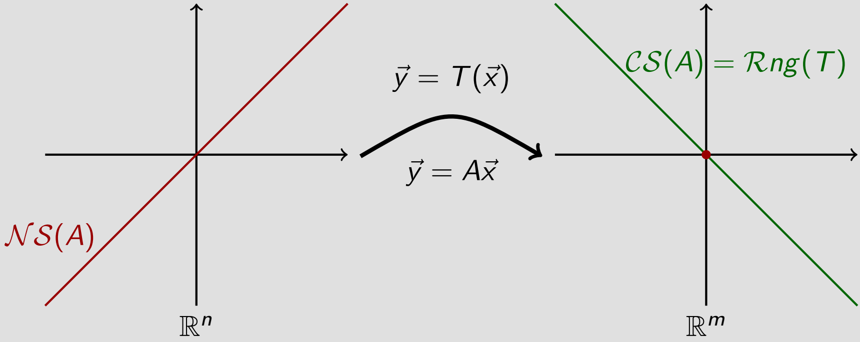

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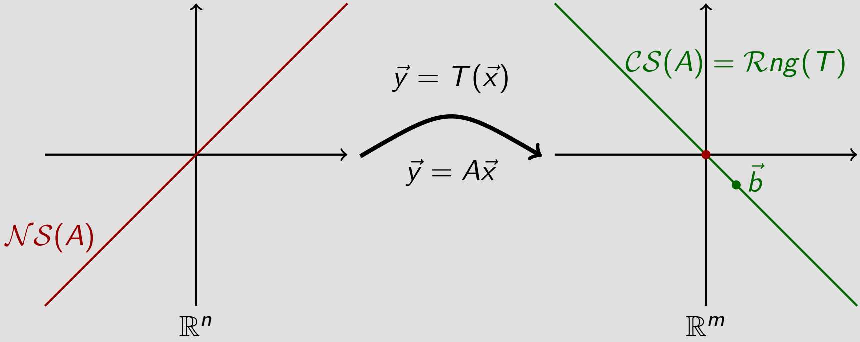

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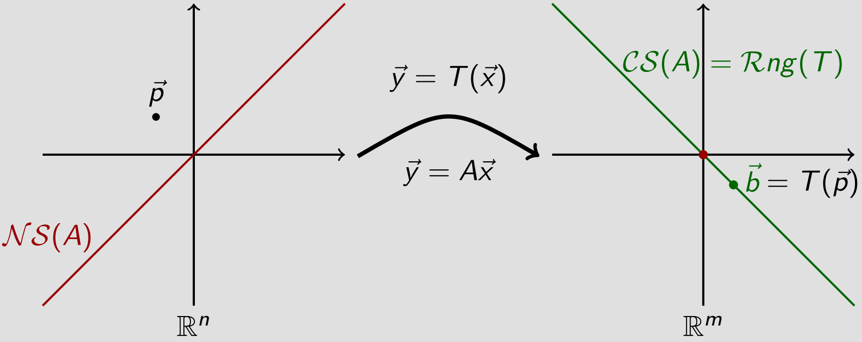

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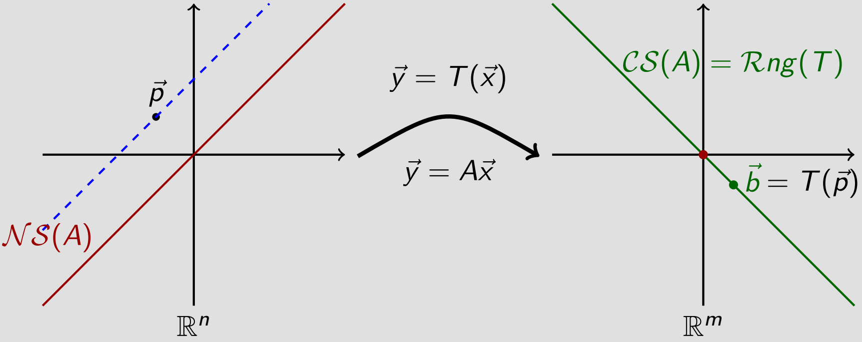

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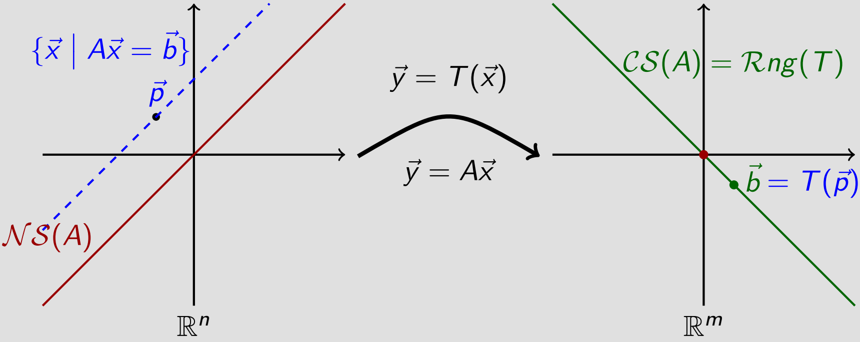

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So, “find” means to find a *linearly independent spanning* set.

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- The LT $\vec{x} \mapsto A\vec{x}$ is one-to-one.

Example—Null Space and Column Space

Find the null space and column space of

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & -1 \end{bmatrix} \text{ and determine when } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ belongs to } \mathcal{CS}(A).$$

Null Space versus Column Space

See the table on page 206 in the textbook!

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Basic Fact about Vector SubSpaces

Let $\mathbb{V}$ be a vector subspace. Suppose $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ are in $\mathbb{V}$.
Then each vector in $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ lies in $\mathbb{V}$.

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