Vector Subspaces of $\mathbb{R}^n$
Column Space & Null Space

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

Vector SubSpace of a Vector Space

Let $\mathbb{V}$ be a vector space. Recall that $\mathbb{W}$ a *vector subspace* of $\mathbb{V}$ if and only if

1. $\vec{0}$ is in $\mathbb{W}$
2. $\mathbb{W}$ *closed wrt vector addition* ($\vec{u}, \vec{v}$ in $\mathbb{W} \implies \vec{u} + \vec{v}$ in $\mathbb{W}$)
3. $\mathbb{W}$ *closed wrt scalar mult* (s scalar, $\vec{v}$ in $\mathbb{W} \implies s\vec{v}$ in $\mathbb{W}$).

Example (Basic Vector SubSpace)

For any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in a vector space $\mathbb{V}$, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is the vector subspace of $\mathbb{V}$ *spanned* by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Basic Fact about Vector SubSpaces

Let $\mathbb{W}$ be a vector subspace of a vector space $\mathbb{V}$. Suppose $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p$ are in $\mathbb{W}$. Then $\mathcal{S}pan\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$ is a vector subspace of $\mathbb{W}$.

Column Space of a Matrix

Let $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ be an $m \times n$ matrix; so, each $\vec{a}_j$ is in $\mathbb{R}^m$.

The *column space* $\mathcal{CS}(A)$ of A is the span of the columns of A, i.e., $\mathcal{CS}(A) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$; this is a vector subspace of $\mathbb{R}^m$.

Three Ways to View $\mathcal{CS}(A)$

The column space $\mathcal{CS}(A)$ of $A = \begin{bmatrix} \vec{a}_1 & \vec{a}_2 & \dots & \vec{a}_n \end{bmatrix}$ is:

- $\mathcal{CS}(A) = \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\}$
- $\mathcal{CS}(A) = \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\}$
- $\mathcal{CS}(A) = \mathcal{R}ng(T)$ where $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

Three Ways to View the Column Space $\mathcal{CS}(A)$

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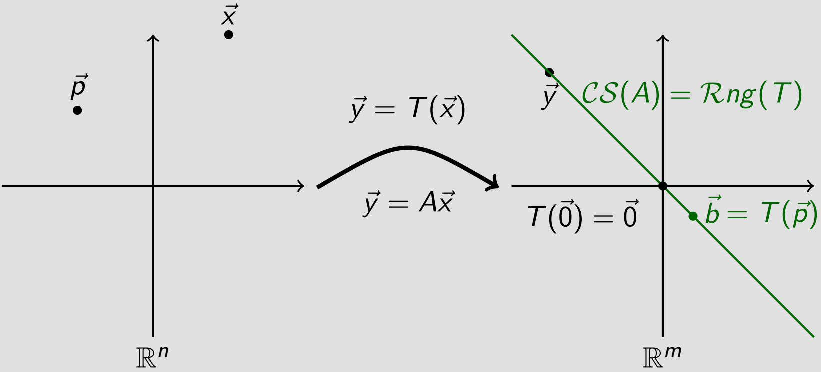
- $\mathcal{CS}(A) = \mathcal{R}ng(T)$ where $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

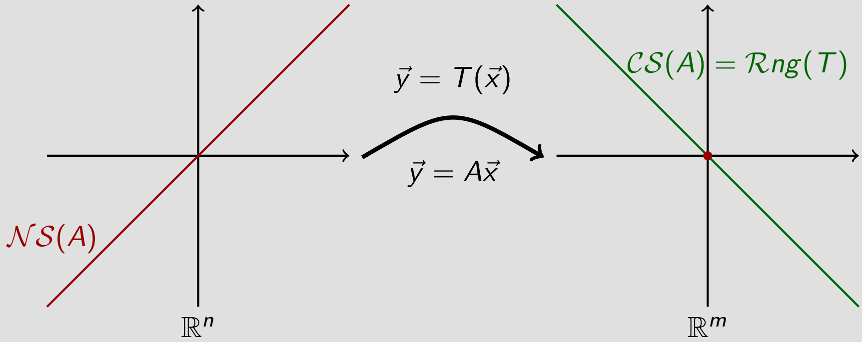


Null Space of a Matrix

Again, let A be an $m \times n$ matrix. The *null space* $\mathcal{NS}(A)$ of A is

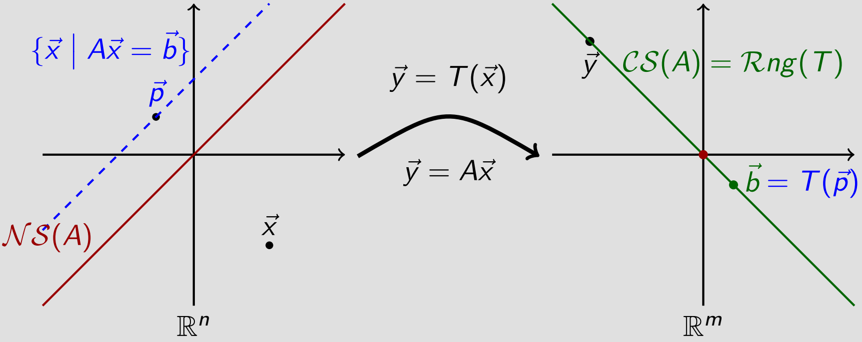
$$\mathcal{NS}(A) = \{\vec{x} \mid A\vec{x} = \vec{0}\}$$

which is the solution set for the homogeneous equation $A\vec{x} = \vec{0}$. This is a vector subspace of $\mathbb{R}^n$.



$A = \left[\vec{a}_1\ \vec{a}_2\ \dots\ \vec{a}_n\right]$ an $m \times n$ matrix and $\mathbb{R}^n \xrightarrow{T} \mathbb{R}^m$ is $T(\vec{x}) = A\vec{x}$

$$\begin{aligned} \mathcal{NS}(A) &= \{\vec{x} \mid A\vec{x} = \vec{0}\} \quad \text{and} \\ \mathcal{CS}(A) &= \mathcal{S}pan\{\vec{a}_1, \vec{a}_2, \dots, \vec{a}_n\} \\ &= \{\vec{b} \text{ in } \mathbb{R}^m \mid A\vec{x} = \vec{b} \text{ has a solution}\} \\ &= \mathcal{R}ng(T) \end{aligned}$$



"Finding" Null Space and Column Space

To "find" the null space $\mathcal{NS}(A)$ and column space $\mathcal{CS}(A)$ of a matrix A:

- row reduce A to E, a REF (or RREF) for A
- columns of E containing row leaders correspond to *pivot* columns of A
- the *pivot* columns of A are LI and span $\mathcal{CS}(A)$
- write the SS for $A\vec{x} = \vec{0}$ in parametric vector form
- identify LI vectors that span $\mathcal{NS}(A)$

So, "find" means to find a *linearly independent spanning* set. Such a set—a *linearly independent spanning set*—is called a *basis*.

More about the Null Space

For any matrix A, these are equivalent:

- $\mathcal{NS}(A) = \{\vec{0}\}$.
- The only solution to $A\vec{x} = \vec{0}$ is $\vec{x} = \vec{0}$.
- The columns of A are LI.
- The LT $\vec{x} \mapsto A\vec{x}$ is one-to-one.

Example—Null Space and Column Space

Find the null space and column space of

$$A = \begin{bmatrix} 1 & -1 & 1 & 2 \\ 1 & 0 & 1 & 1 \\ 1 & 2 & 1 & -1 \end{bmatrix} \text{ and determine when } \vec{b} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} \text{ belongs to } \mathcal{CS}(A).$$

Null Space versus Column Space

Please study the table on page 206 in the textbook! This is full of excellent info, and you should understand all of what is written there.

Also, please review **all** that we did back in Chapter 2, Section 8. There are four videos, but you can review by looking at the associated pdfs.