Vector Spaces and SubSpaces

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

What is a Vector Space?

A *vector space* is a "bunch" of objects—that we call *vectors*—with the properties that we can add any two vectors and we can multiply any vector by any scalar.

Let $\mathbb{V}$ be a set. Suppose we have a way of

- adding any two elements of $\mathbb{V}$
- multiplying any element of $\mathbb{V}$ by any scalar

That is,

- given $\vec{v}$ and $\vec{w}$ in $\mathbb{V}$, there is a $\vec{v} + \vec{w}$ in $\mathbb{V}$
- given $\vec{v}$ in $\mathbb{V}$ and any scalar s, there is a $s\vec{v}$ in $\mathbb{V}$

Then we call $\mathbb{V}$ a *vector space*, provided certain axioms hold.

Examples of Vector Spaces

Some simple examples:

1. $\mathbb{R}^n$ is a vector space $\ddot\smile$
2. $\mathbb{R}^{m\times n}$ is the vector space of all $m \times n$ matrices (given $m \times n$ matrices A and B, we know what $A + B$ and sA are, right?)
3. $\mathbb{C}^n$ is a vector space (here the coordinates are complex numbers)
4. Any vector subspace of $\mathbb{R}^n$ is itself a vector space, right?
5. $\mathbb{R}^\infty = \{(x_n)_{n=1}^\infty\}$ is the vector space of all sequences (of real numbers)

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With these ways of adding and multiplying by scalars, the family

$$\mathbb{F}(\mathcal{X} \to \mathbb{R}) = \big\{\text{all } \boldsymbol{f} \text{ with } \mathcal{X} \xrightarrow{\boldsymbol{f}} \mathbb{R}\big\}$$

of real-valued functions on $\mathcal{X}$ becomes a vector space.

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Basic Fact about Vector SubSpaces

Let $\mathbb{W}$ be a vector subspace of a vector space $\mathbb{V}$. Suppose $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p$ are in $\mathbb{W}$. Then $\mathcal{S}pan\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$ lies in $\mathbb{W}$.

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Homework: Explain why this fact is true.

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Example (Basic Vector SubSpace)

For any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in a vector space $\mathbb{V}$, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is the vector subspace of $\mathbb{V}$ *spanned* by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

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Can you find spanning sets for the vector subspaces?

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- $\{$all continuous $\boldsymbol{f}$ in $\mathbb{F}\}$.
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Of course, $\mathbb{P}$ is also a vector space all by itself.

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Can you find spanning sets for the vector subspaces?