Vector Spaces and SubSpaces

Linear Algebra
MATH 2076

What is a Vector Space?

A *vector space* is a "bunch" of objects—that we call *vectors*—with the properties that we can add any two vectors and we can multiply any vector by any scalar.

Let $\mathbb{V}$ be a set. Suppose we have a way of

- adding any two elements of $\mathbb{V}$
- multiplying any element of $\mathbb{V}$ by any scalar

That is,

- given $\vec{v}$ and $\vec{w}$ in $\mathbb{V}$, there is a $\vec{v} + \vec{w}$ in $\mathbb{V}$
- given $\vec{v}$ in $\mathbb{V}$ and any scalar s, there is a $s\vec{v}$ in $\mathbb{V}$

Then we call $\mathbb{V}$ a *vector space*, provided certain axioms hold.

Examples of Vector Spaces

Some simple examples:

1. $\mathbb{R}^n$ is a vector space $\ddot\smile$
2. $\mathbb{R}^{m \times n}$ is the vector space of all $m \times n$ matrices (given $m \times n$ matrices A and B, we know what $A + B$ and sA are, right?)
3. $\mathbb{C}^n$ is a vector space (here the coordinates are complex numbers)
4. Any vector subspace of $\mathbb{R}^n$ is itself a vector space, right?
5. $\mathbb{R}^\infty = \{(x_n)_{n=1}^\infty\}$ is the vector space of all sequences (of real numbers)

Fundamental Example of a Vector Space

Let $\mathcal{X}$ be any set, and let $\mathbb{F}(\mathcal{X} \to \mathbb{R})$ be the family of all functions with domain $\mathcal{X}$ and codomain $\mathbb{R}$.

Thus each object $\boldsymbol{f}$ in $\mathbb{F}$ is a function $\mathcal{X} \xrightarrow{\boldsymbol{f}} \mathbb{R}$.
We define $\boldsymbol{f} + \boldsymbol{g}$ by the formula

for each x in $\mathcal{X}$, $\big(\boldsymbol{f} + \boldsymbol{g}\big)(x) := \boldsymbol{f}(x) + \boldsymbol{g}(x)$

and then define $s\boldsymbol{f}$ by the formula

for each x in $\mathcal{X}$, $\big(s\boldsymbol{f}\big)(x) := s\boldsymbol{f}(x)$.

With these ways of adding and multiplying by scalars, the family

$$\mathbb{F}(\mathcal{X} \to \mathbb{R}) = \big\{\text{all } \boldsymbol{f} \text{ with } \mathcal{X} \xrightarrow{\boldsymbol{f}} \mathbb{R}\big\}$$

of real-valued functions on $\mathcal{X}$ becomes a vector space.

What is a SubSpace of a Vector Space?

Let $\mathbb{V}$ be a vector space. We call $\mathbb{W}$ a *vector subspace* of $\mathbb{V}$ if and only if

1. $\vec{0}$ is in $\mathbb{W}$,
2. $\mathbb{W}$ *closed wrt vector addition*, ($\vec{u}, \vec{v}$ in $\mathbb{W} \implies \vec{u} + \vec{v}$ in $\mathbb{W}$),
3. $\mathbb{W}$ *closed wrt scalar mult* (s scalar , $\vec{v}$ in $\mathbb{W} \implies s\vec{v}$ in $\mathbb{W}$).

Note that

- (3) says that whenever $\vec{w}$ is in $\mathbb{W}$, $\mathcal{S}pan\{\vec{w}\}$ lies in $\mathbb{W}$, and
- (2) and (3) together say that any LC of vectors in $\mathbb{W}$ is in $\mathbb{W}$.

Basic Fact about Vector SubSpaces

Let $\mathbb{W}$ be a vector subspace of a vector space $\mathbb{V}$. Suppose $\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p$ are in $\mathbb{W}$. Then $\mathcal{S}pan\{\vec{w}_1, \vec{w}_2, \ldots, \vec{w}_p\}$ lies in $\mathbb{W}$.

Homework: Explain why this fact is true.

Examples of Vector SubSpaces of a Vector Space?

Let $\mathbb{V}$ be a vector space. Here are some simple examples of vector subspaces of $\mathbb{V}$.

- $\mathbb{W} = \{\vec{0}\}$ is the *trivial* vector subspace
- $\mathbb{W} = \mathbb{V}$ is a vector subspace of itself (also kinda *trivial*)
- $\mathbb{W} = \mathcal{S}pan\{\vec{v}\}$ (for any $\vec{v}$ in $\mathbb{V}$)
- $\mathbb{W} = \mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ (for any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in $\mathbb{V}$)

In the last example above, we call $\mathbb{W}$ the *subspace spanned by* $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$, and then $\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is called a *spanning set* for $\mathbb{W}$.

Example (Basic Vector SubSpace)

For any $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$ in a vector space $\mathbb{V}$, $\mathcal{S}pan\{\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p\}$ is the vector subspace of $\mathbb{V}$ *spanned* by $\vec{v}_1, \vec{v}_2, \ldots, \vec{v}_p$.

Vector Subspaces of $\mathbb{R}^{m \times n}$

Recall that $\mathbb{R}^{m \times n}$ is the vector space of all $m \times n$ matrices. Which of the following are vectors subspaces of $\mathbb{R}^{n \times n}$? If not, why not?

- All upper triangular $n \times n$ matrices.
- All invertible $n \times n$ matrices.
- All $n \times n$ matrices A with $\det(A) = 1$.
- All $n \times n$ matrices A with $\det(A) = 0$.
- All $n \times n$ matrices A with $A^T = A$ (the *symmetric* matrices).
- All $n \times n$ matrices A with $A^T = -A$ (the *skew-symmetric* matrices).

Can you find spanning sets for the vector subspaces?

Function Spaces

Let $\mathbb{F} = \mathbb{F}(\mathbb{R} \to \mathbb{R}) = \{\text{all } \boldsymbol{f} \text{ with } \mathbb{R} \xrightarrow{\boldsymbol{f}} \mathbb{R}\}$. This is a vector space with the usual ways of adding and multiplying by scalars:

- $\boldsymbol{f} + \boldsymbol{g}$ is defined by

  for each t in $\mathbb{R}$, $\big(\boldsymbol{f} + \boldsymbol{g}\big)(t) := \boldsymbol{f}(t) + \boldsymbol{g}(t)$
- and $s\boldsymbol{f}$ is defined by

  for each t in $\mathbb{R}$, $\big(s\boldsymbol{f}\big)(t) := s\boldsymbol{f}(t)$.

Notice that we use t for our variable instead of x.

Here are some important vector subspaces of $\mathbb{F}$.

- $\{$all continuous $\boldsymbol{f}$ in $\mathbb{F}\}$.
- $\{$all differentiable $\boldsymbol{f}$ in $\mathbb{F}\}$.
- $\{$all twice differentiable $\boldsymbol{f}$ in $\mathbb{F}\}$.
- $\mathcal{S}pan\{e^t\}$ (the solution set to $y' = y$)
- $\mathcal{S}pan\{e^t, e^{-t}\}$ (the solution set to $y'' = y$)
- $\mathcal{S}pan\{\cos(t), \sin(t)\}$ (the solution set to $y'' + y = 0$)

The Space of all Polynomials

Let $\mathbb{P}$ be the family of all *polynomials*. Thus $\boldsymbol{p}$ is in $\mathbb{P}$ if and only if $\boldsymbol{p}$ is a function of the form

$$\boldsymbol{p}(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n.$$

Here $c_0, c_1, \ldots, c_n$ are constants, called the *coefficients* of the polynomial $\boldsymbol{p}$, and when $c_n \neq 0$ we say that $\boldsymbol{p}$ has *degree n*. (By definition, the zero polynomial has degree zero.)

Clearly the sum $\boldsymbol{p} + \boldsymbol{q}$ of two polys $\boldsymbol{p}$ and $\boldsymbol{q}$ is again a poly, as is any scalar multiple $s\boldsymbol{p}$. Thus, $\mathbb{P}$ is a vector subspace of $\mathbb{F}$.

Of course, $\mathbb{P}$ is also a vector space all by itself.

Subspaces of Polynomials

Let $\mathbb{P}$ be the family of all *polynomials*. Which of the following are vector subspaces of $\mathbb{P}$? If not, why not?

- All polynomials of degree n.
- All polynomials of degree n or less. (We call this subspace $\mathbb{P}_n$.)
- All polynomials of even degree.
- All polynomials of odd degree.
- All polynomials $\boldsymbol{p}$ with $\boldsymbol{p}(0) = 1$.
- All polynomials $\boldsymbol{p}$ with $\boldsymbol{p}(0) = 0$.

Can you find spanning sets for the vector subspaces?