Vector Spaces—An Introduction

Linear Algebra
MATH 2076

UNIVERSITY OF Cincinnati

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Then we call $\mathbb{V}$ a *vector space*, provided certain axioms hold.

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5. $\mathbb{R}^\infty = \{(x_n)_{n=1}^\infty\}$ is the vector space of all sequences (of real numbers)

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E.g., look at $3\sin(x) + 2\cos(5x)$.

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Here $c_0, c_1, \ldots, c_n$ are constants, called the *coefficients* of the polynomial $\boldsymbol{p}$, and when $c_n \neq 0$ we say that $\boldsymbol{p}$ has *degree* n.

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