Vector Spaces—An Introduction

Linear Algebra
MATH 2076

What is a Vector Space?

A *vector space* is a "bunch" of objects—that we call *vectors*—with the property that we can add any two vectors and we can multiply any vector by any scalar.

Let $\mathbb{V}$ be a set. Suppose we have a way of

- adding any two elements of $\mathbb{V}$
- multiplying any element of $\mathbb{V}$ by any scalar

That is,

- given $\vec{v}$ and $\vec{w}$ in $\mathbb{V}$, there is a $\vec{v} + \vec{w}$ in $\mathbb{V}$
- given $\vec{v}$ in $\mathbb{V}$ and any scalar s, there is a $s\vec{v}$ in $\mathbb{V}$

Then we call $\mathbb{V}$ a *vector space*, provided certain axioms hold.

Examples of Vector Spaces

Some simple examples:

1. $\mathbb{R}^n$ is a vector space $\ddot{\smile}$
2. $\mathbb{R}^{m\times n}$ is the vector space of all $m \times n$ matrices (given $m \times n$ matrices A and B, we know what $A + B$ and sA are, right?)
3. $\mathbb{C}^n$ is a vector space (here the coordinates are complex numbers)
4. Any vector subspace of $\mathbb{R}^n$ is itself a vector space, right?
5. $\mathbb{R}^\infty = \{(x_n)_{n=1}^\infty\}$ is the vector space of all sequences (of real numbers)

Fundamental Example of a Vector Space

Let $\mathcal{X}$ be any set, and let $\mathbb{F}$ be the family of all functions with domain $\mathcal{X}$ and codomain $\mathbb{R}$.

Thus each object $\boldsymbol{f}$ in $\mathbb{F}$ is a function $\mathcal{X} \xrightarrow{\boldsymbol{f}} \mathbb{R}$.

Now let $\boldsymbol{f}$ and $\boldsymbol{g}$ be elements of $\mathbb{F}$. Can we: add? multiply by scalars?

Let's define $\boldsymbol{f} + \boldsymbol{g}$ by the formula

for each x in $\mathcal{X}$, $\big(\boldsymbol{f} + \boldsymbol{g}\big)(x) := \boldsymbol{f}(x) + \boldsymbol{g}(x)$

and then define $s\boldsymbol{f}$ by the formula

for each x in $\mathcal{X}$, $\big(s\boldsymbol{f}\big)(x) := s\boldsymbol{f}(x)$.

With these ways of adding and multiplying by scalars, the family $\mathbb{F}$ of real-valued functions on $\mathcal{X}$ becomes a vector space.

E.g., look at $3\sin(x) + 2\cos(5x)$.

Polynomials

Let $\mathbb{P}$ be the family of all *polynomials*. Thus $\boldsymbol{p}$ is in $\mathbb{P}$ if and only if $\boldsymbol{p}$ is a function of the form

$$\boldsymbol{p}(t) = c_0 + c_1 t + c_2 t^2 + \cdots + c_n t^n.$$

Here $c_0, c_1, \ldots, c_n$ are constants, called the *coefficients* of the polynomial $\boldsymbol{p}$, and when $c_n \neq 0$ we say that $\boldsymbol{p}$ has *degree n*.

Clearly the sum $\boldsymbol{p} + \boldsymbol{q}$ of two polys $\boldsymbol{p}$ and $\boldsymbol{q}$ is again a poly, as is any scalar multiple $s\boldsymbol{p}$. Thus, $\mathbb{P}$ is a vector space.