A Hyperplane in $\mathbb{P}_3$

Linear Algebra
MATH 2076

The Problem and Our Strategy

Let $\mathbb{W}$ be the space of all polynomials $\boldsymbol{p}$ in $\mathbb{P}_3$ that satisfy $\boldsymbol{p}(2) = 0$.
Here we:

- Explain why $\mathbb{W}$ is a vector subspace of $\mathbb{P}_3$.
- Find a basis $\mathcal{B}$ for $\mathbb{W}$ and determine the dimension of $\mathbb{W}$.
- Find the $\mathcal{B}$-coordinate vector for $\boldsymbol{p}(t) = (t-1)(t-2)(t-3)$.

We make extensive use of coordinate vectors, so you might want to review this material. In particular, remember that:

- vectors are LI if and only if their coordinate vectors are LI, and,
- coordinate maps "preserve" all linear combinations.

Finding Some Vectors in $\mathbb{W}$

It is straightforward to check that $\mathbb{W}$ is a vector subspace of $\mathbb{P}_3$.

Since the polynomial $\boldsymbol{q}_1$, given by $\boldsymbol{q}_1(t) = t - 2$, belongs to $\mathbb{W}$, $\mathbb{W}$ is not the zero subspace. Also, $\mathbb{W} \neq \mathbb{P}_3$ (for example, because $\boldsymbol{t}^3$ is not in $\mathbb{W}$).

Since $\{\vec{0}\} \neq \mathbb{W} \neq \mathbb{P}_3$, $1 \leq \dim \mathbb{W} \leq 3$.

Let's find a few more polynomials in $\mathbb{W}$. Of course, $s\boldsymbol{q}_1$ is in $\mathbb{W}$ for any scalar s, but we want more than just scalar multiples.

Notice that $\boldsymbol{q}_2 = \boldsymbol{t} \cdot \boldsymbol{q}_1$ and $\boldsymbol{q}_3 = \boldsymbol{t}^2 \cdot \boldsymbol{q}_1$ both belong to $\mathbb{W}$.

So, we have three polynomials, $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$, that are all in $\mathbb{W}$. Are these LI?

$\mathbb{W}$ is all polynomials $\boldsymbol{p}$ in $\mathbb{P}_3$ that satisfy $\boldsymbol{p}(2) = 0$

Finding a Basis for $\mathbb{W}$

We have three polynomials, $\boldsymbol{q}_1, \boldsymbol{q}_2 = \boldsymbol{t} \cdot \boldsymbol{q}_1, \boldsymbol{q}_3 = \boldsymbol{t}^2 \cdot \boldsymbol{q}_1$ that are all in $\mathbb{W}$. Here

$$\begin{aligned} \boldsymbol{q}_1(t) &= t - 2 = -2 + t \\ \boldsymbol{q}_2(t) &= t(t-2) = -2t + t^2 \\ \boldsymbol{q}_3(t) &= t^2(t-2) = -2t^2 + t^3. \end{aligned}$$

If these are LI, then they form a basis. Right? Why?

To check for LI, we could look at $s_1\boldsymbol{q}_1 + s_2\boldsymbol{q}_2 + s_3\boldsymbol{q}_3 = \mathbf{0}$ and explain why this means that $s_1 = s_2 = s_3 = 0$.

Instead, we use the fact that vectors are LI iff their coord vectors are LI.

To employ this strategy, we need a basis for $\mathbb{P}_3$. We use the standard basis, $\mathcal{P} = \{\mathbf{1}, \boldsymbol{t}, \boldsymbol{t}^2, \boldsymbol{t}^3\}$.

$\mathbb{W}$ is all polynomials $\boldsymbol{p}$ in $\mathbb{P}_3$ that satisfy $\boldsymbol{p}(2) = 0$

Finding a Basis for $\mathbb{W}$

The three polynomials $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ are in $\mathbb{W}$; here

$$\begin{aligned} \boldsymbol{q}_1(t) &= t - 2 = -2 + t \\ \boldsymbol{q}_2(t) &= t(t-2) = -2t + t^2 \\ \boldsymbol{q}_3(t) &= t^2(t-2) = -2t^2 + t^3. \end{aligned}$$

Using the standard basis, $\mathcal{P} = \{\mathbf{1}, \boldsymbol{t}, \boldsymbol{t^2}, \boldsymbol{t^3}\}$, we see that the $\mathcal{P}$-coordinate vectors for $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ are

$$[\boldsymbol{q}_1]_{\mathcal{P}} = \begin{bmatrix} -2 \\ 1 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad [\boldsymbol{q}_2]_{\mathcal{P}} = \begin{bmatrix} 0 \\ -2 \\ 1 \\ 0 \end{bmatrix} \quad \text{and} \quad [\boldsymbol{q}_3]_{\mathcal{P}} = \begin{bmatrix} 0 \\ 0 \\ -2 \\ 1 \end{bmatrix}.$$

How do we test these for linear independence? They're just vectors in $\mathbb{R}^4$.

We look at the matrix $\begin{bmatrix} [\boldsymbol{q}_1]_{\mathcal{P}} & [\boldsymbol{q}_2]_{\mathcal{P}} & [\boldsymbol{q}_3]_{\mathcal{P}} \end{bmatrix}$.

$\mathbb{W}$ is all polynomials $\boldsymbol{p}$ in $\mathbb{P}_3$ that satisfy $\boldsymbol{p}(2) = 0$

Finding a Basis for $\mathbb{W}$ and Determining dim $\mathbb{W}$

The polynomials $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ are LI iff $[\boldsymbol{q}_1]_{\mathcal{P}}, [\boldsymbol{q}_2]_{\mathcal{P}}, [\boldsymbol{q}_3]_{\mathcal{P}}$ are LI. To see if these $\mathcal{P}$-coordinate vectors are LI, we look at the matrix

$$\left[[\boldsymbol{q}_1]_{\mathcal{P}} \; [\boldsymbol{q}_2]_{\mathcal{P}} \; [\boldsymbol{q}_3]_{\mathcal{P}} \right] = \begin{bmatrix} -2 & 0 & 0 \\ 1 & -2 & 0 \\ 0 & 1 & -2 \\ 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}$$

which has the indicated REF. Thus $\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3$ are LI, so $\mathcal{B} = \{\boldsymbol{q}_1, \boldsymbol{q}_2, \boldsymbol{q}_3\}$ is a basis for $\mathbb{W}$.

Therefore, dim $\mathbb{W} = 3$; $\mathbb{W}$ is a 3-plane (aka, a hyperplane) in $\mathbb{P}_3$.

$\mathbb{W}$ is all polynomials $\boldsymbol{p}$ in $\mathbb{P}_3$ that satisfy $\boldsymbol{p}(2) = 0$

Finding the $\mathcal{B}$-coordinate Vector for $\boldsymbol{p}$

The polynomial $\boldsymbol{p}$, given by $\boldsymbol{p}(t) = (t-1)(t-2)(t-3)$, has $\boldsymbol{p}(2) = 0$, so $\boldsymbol{p}$ is a vector in $\mathbb{W}$. Therefore, we can write $\boldsymbol{p} = c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3$, and c_1, c_2, c_3 are the $\mathcal{B}$-coordinates for $\boldsymbol{p}$.

How do we find these coordinates?

One way is to use the fact that coordinate vectors preserve LCs. Right? This means that $\boldsymbol{p} = c_1\boldsymbol{q}_1 + c_2\boldsymbol{q}_2 + c_3\boldsymbol{q}_3$ iff

$$[\boldsymbol{p}]_{\mathcal{P}} = c_1[\boldsymbol{q}_1]_{\mathcal{P}} + c_2[\boldsymbol{q}_2]_{\mathcal{P}} + c_3[\boldsymbol{q}_3]_{\mathcal{P}}.$$

Now we're looking at vectors in $\mathbb{R}^4$, and we know how to solve this vector equation. Right?

$\mathbb{W}$ is all polynomials $\boldsymbol{p}$ in $\mathbb{P}_3$ that satisfy $\boldsymbol{p}(2) = 0$

Finding the $\mathcal{B}$-coordinate Vector for $\boldsymbol{p}$

We gotta solve $c_1\left[\boldsymbol{q}_1\right]_{\mathcal{P}} + c_2\left[\boldsymbol{q}_2\right]_{\mathcal{P}} + c_3\left[\boldsymbol{q}_3\right]_{\mathcal{P}} = \left[\boldsymbol{p}\right]_{\mathcal{P}}$.

Here $\boldsymbol{p}(t) = (t-1)(t-2)(t-3) = -6 + 11t - 6t^2 + t^3$, so $\left[\boldsymbol{p}\right]_{\mathcal{P}} = \begin{bmatrix} -6 \\ 11 \\ -6 \\ 1 \end{bmatrix}$.

Remembering what $\left[\boldsymbol{q}_1\right]_{\mathcal{P}}, \left[\boldsymbol{q}_2\right]_{\mathcal{P}}, \left[\boldsymbol{q}_3\right]_{\mathcal{P}}$ are, we get the augmented matrix

$$\left[\begin{array}{ccc|c} -2 & 0 & 0 & -6 \\ 1 & -2 & 0 & 11 \\ 0 & 1 & -2 & -6 \\ 0 & 0 & 1 & 1 \end{array}\right] \sim \left[\begin{array}{ccc|c} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & -4 \\ 0 & 0 & 1 & 1 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

and then some elem row ops produce the indicated REF. Right? So,

$c_1 = 3, c_2 = -4, c_3 = 1$ and therefore $\left[\boldsymbol{p}\right]_{\mathcal{B}} = \begin{bmatrix} 3 \\ -4 \\ 1 \end{bmatrix}$.

$\mathbb{W}$ is all polynomials $\boldsymbol{p}$ in $\mathbb{P}_3$ that satisfy $\boldsymbol{p}(2) = 0$